Lecture 11
Polar Coordinates and Polar Equations

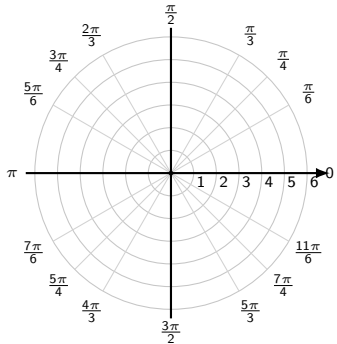
Pitambar Acharya

November 5, 2026

Our Flight Plan

1. Plot points using polar coordinates
2. Switch between polar and rectangular addresses
3. Translate entire equations from one language to the other
4. Sketch lines, circles, cardioids, roses and spirals

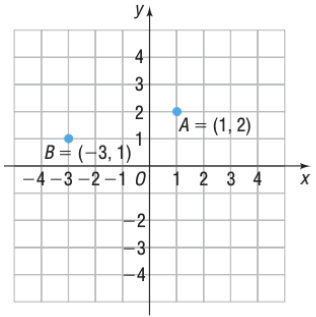
Blue text = worked solution.



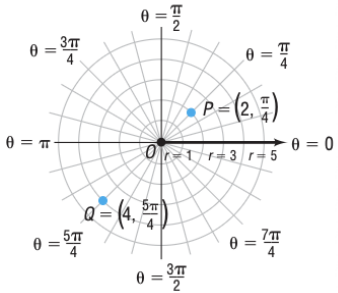
Part I

Polar Coordinates

Two Ways to Say "Right Here"



(a) Rectangular grid



(b) Polar grid

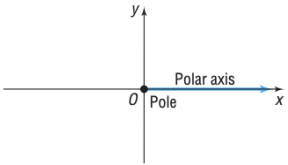
Rectangular: *how far over, how far up.* Polar: *how far out, which direction.*

Anatomy of a Polar Grid

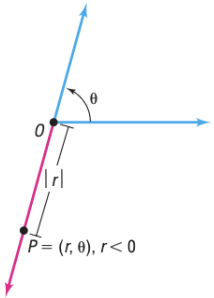
Choose a home base, the **pole**, and a ray leaving it, the **polar axis**.

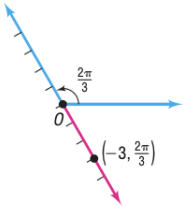
A point P gets the address (r, θ):

- r: how far P sits from the pole (for $r > 0$)
- θ: how much you rotate the polar axis to aim at P (degrees or radians)



Walking Backwards: Negative r





If $r < 0$, first aim along angle θ, then walk $|r|$ units in the **opposite** direction.

- $\theta > 0$: rotate counterclockwise
- $\theta < 0$: rotate clockwise
- $r = 0$: you are at the pole, whatever θ is

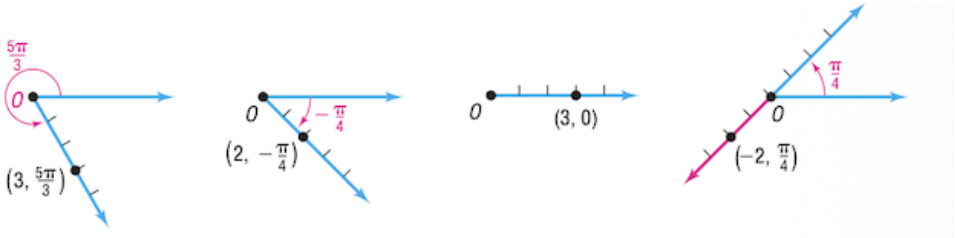
Example 1: Drop the Pins

Plot the following polar points:

$$A\left(3, \tfrac{5\pi}{3}\right), \quad B\left(2, -\tfrac{\pi}{4}\right), \quad C(3, 0), \quad D\left(-2, \tfrac{\pi}{4}\right)$$

Special attention: a negative angle spins clockwise; a negative r reverses the walk.

Solution.



One Point, Several Polar Coordinates

For any integer k, all of these name the *same* point:

$$(r, \theta) = (r, \theta + 2\pi k) = (-r, \theta + \pi + 2\pi k)$$

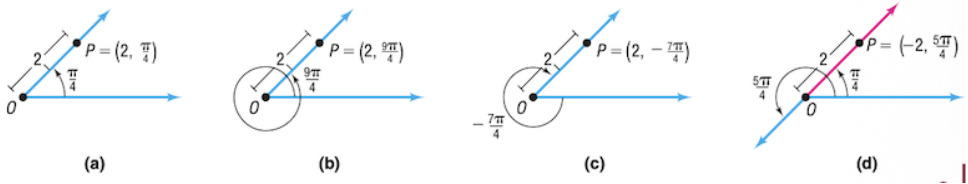
and the pole is $(0, \theta)$ for *every* angle θ.

Special attention: $+2\pi$ is one full revolution; $+\pi$ is a half-turn, which is why the sign of r must flip.

Example 2: Finding Several Polar Coordinates of a Single Point

Give three more addresses for $\left(2, \frac{\pi}{4}\right)$: two with $r > 0$ and one with $r < 0$.

Solution. Add or subtract full revolutions when $r > 0$; add π and flip the sign of r when $r < 0$.

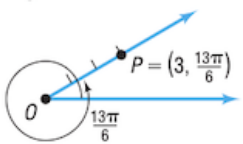


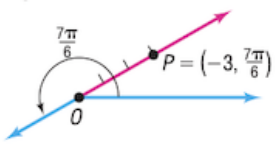
Example 3: Addresses on Demand

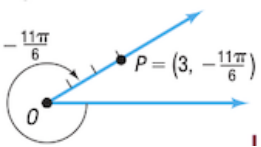
Point $P = \left(3, \frac{\pi}{6}\right)$. Give another address for P that satisfies:
(a) $r > 0,\ 2\pi \le \theta < 4\pi$ (b) $r < 0,\ 0 \le \theta < 2\pi$ (c) $r > 0,\ -2\pi \le \theta < 0$

Solution.

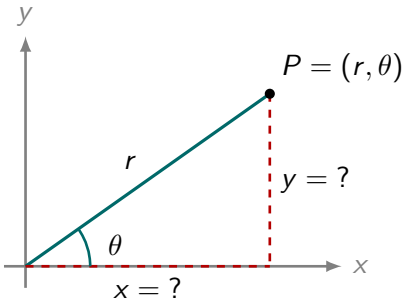
(a) one extra revolution: $\left(3, \frac{\pi}{6} + 2\pi\right) = \left(3, \frac{13\pi}{6}\right)$
(b) flip r, add π: $\left(-3, \frac{\pi}{6} + \pi\right) = \left(-3, \frac{7\pi}{6}\right)$
(c) back one revolution: $\left(3, \frac{\pi}{6} - 2\pi\right) = \left(3, -\frac{11\pi}{6}\right)$







Polar $\to$ Rectangular: The Right Triangle Trick



Drop a perpendicular from P to the x-axis and read the triangle:

$$\boxed{x = r\cos\theta \qquad y = r\sin\theta}$$

These work for *every* choice of r and θ, even negative ones.

Example 4: Polar Address $\rightarrow (x, y)$

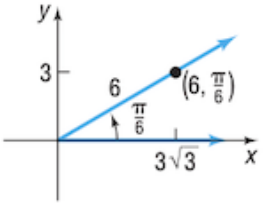
Convert to rectangular coordinates:

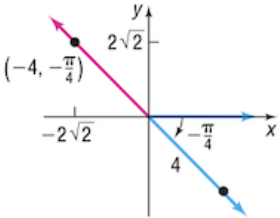
$$\text{(a) } \left(6, \tfrac{\pi}{6}\right) \qquad \text{(b) } \left(-4, -\tfrac{\pi}{4}\right)$$

Solution.

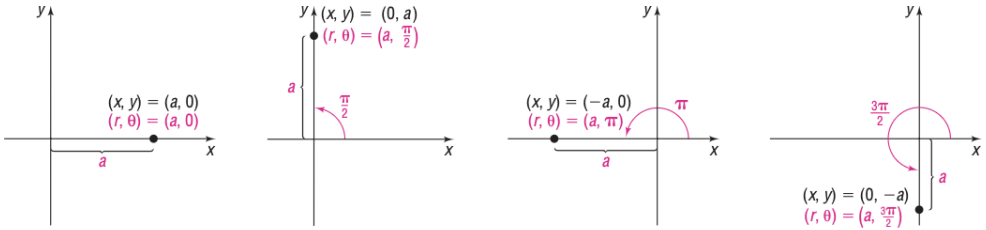
$$\text{(a) } \begin{aligned} x &= 6\cos\tfrac{\pi}{6} = 3\sqrt{3} \\ y &= 6\sin\tfrac{\pi}{6} = 3 \end{aligned} \quad \Rightarrow \ (3\sqrt{3},\, 3)$$

$$\text{(b) } \begin{aligned} x &= -4\cos\left(-\tfrac{\pi}{4}\right) = -2\sqrt{2} \\ y &= -4\sin\left(-\tfrac{\pi}{4}\right) = 2\sqrt{2} \end{aligned} \quad \Rightarrow \ (-2\sqrt{2},\, 2\sqrt{2})$$





Rectangular to Polar



(a) $(x, y) = (a, 0), a > 0$ **(b)** $(x, y) = (0, a), a > 0$ **(c)** $(x, y) = (-a, 0), a > 0$ **(d)** $(x, y) = (0, -a), a > 0$

For $a > 0$, $(0, a)$ sits straight up and has the address $(a, \frac{\pi}{2})$; $(-a, 0)$ has (a, π); $(0, -a)$ has $(a, \frac{3\pi}{2})$.

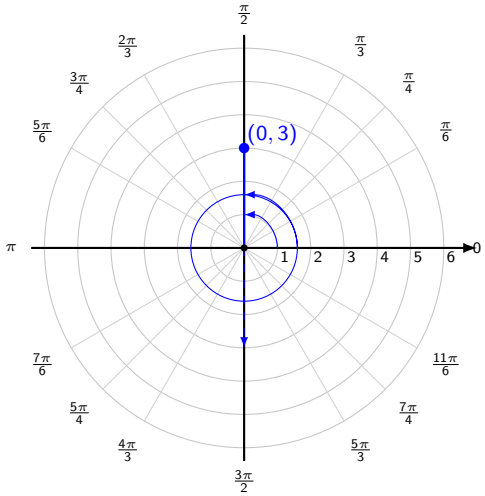
Example 5: An Axis Point

Write $(x, y) = (0, 3)$ in polar form, then give two other names.

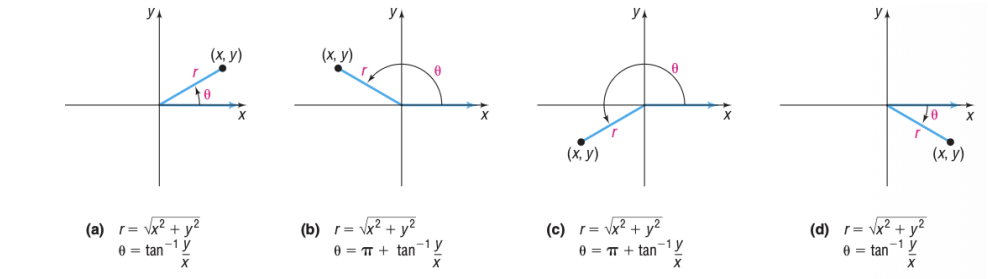
Solution. The point lies 3 units straight up the y-axis.

$$r = 3,\ \theta = \tfrac{\pi}{2} \ \Rightarrow\ \left(3, \tfrac{\pi}{2}\right)$$

Others: $\left(-3, -\tfrac{\pi}{2}\right)$ (aim down, walk backwards) and $\left(3, \tfrac{5\pi}{2}\right)$ (one extra revolution).



Rectangular $\to$ Polar: Choose the Right Quadrant



$r = \sqrt{x^2 + y^2}$ $\quad$ $\theta = \tan^{-1}\frac{y}{x}$ (quadrant I or IV), $\quad$ $\theta = \pi + \tan^{-1}\frac{y}{x}$ (quadrant II or III)

Special attention: your calculator's $\tan^{-1}$ only returns angles in quadrants I and IV, so sketch first!

Example 6: Quadrant IV

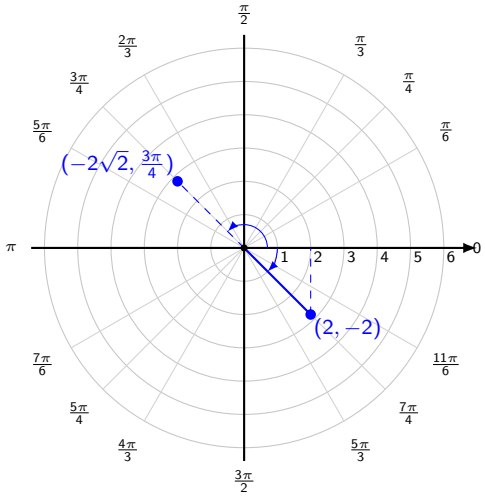
Find polar coordinates of $(2, -2)$ with $r > 0$, $0 \leq \theta < 2\pi$, and then two more names.

Solution.

$$r = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}, \qquad \tan\theta = \frac{-2}{2} = -1$$

Quadrant IV: $\theta = \tan^{-1}(-1) = -\frac{\pi}{4}$, or $\frac{7\pi}{4}$.

$$\left(2\sqrt{2}, -\tfrac{\pi}{4}\right) = \left(2\sqrt{2}, \tfrac{7\pi}{4}\right)$$
$$= \left(-2\sqrt{2}, \tfrac{3\pi}{4}\right)$$



Example 7: Quadrant III

Convert $(-1, -\sqrt{3})$ to polar coordinates, then find two more names.

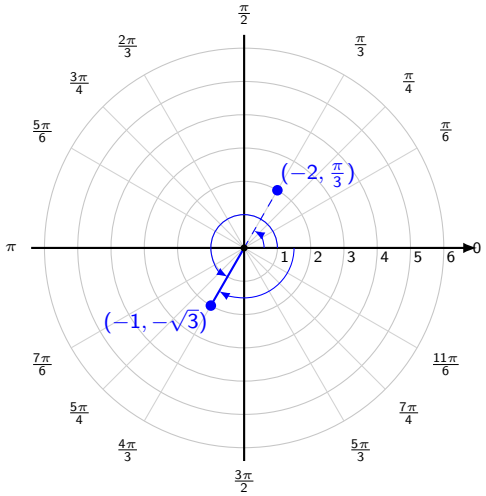
Solution.

$$r = \sqrt{1+3} = 2, \qquad \tan\theta = \sqrt{3}$$

Quadrant III: $\theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$.

$$\begin{aligned} &\left(2, \tfrac{4\pi}{3}\right) = \left(-2, \tfrac{\pi}{3}\right) \\ &= \left(2, -\tfrac{2\pi}{3}\right) \end{aligned}$$

Special attention: $\frac{4\pi}{3} - \pi = \frac{\pi}{3}$ and $\frac{4\pi}{3} - 2\pi = -\frac{2\pi}{3}$.



Equations Have Two Languages Too

To translate an equation, hunt for these swaps:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad x^2 + y^2 = r^2, \qquad \tan\theta = \frac{y}{x}$$

Special attention: multiplying both sides by r is the classic move that turns lonely $\cos\theta$ or $\sin\theta$ into x or y.

Example 8: A Polar Equation $\rightarrow$ Rectangular

Convert $r = 6\cos\theta$ and identify the curve.

Solution. Multiply by r, then swap:

$$r^2 = 6r\cos\theta \;\Rightarrow\; x^2 + y^2 = 6x \;\Rightarrow\; x^2 - 6x + 9 + y^2 = 9$$

$$\boxed{(x-3)^2 + y^2 = 3^2}$$

A circle with center $(3, 0)$ and radius 3.

Example 9: Rectangular $\rightarrow$ Polar

Rewrite $4xy = 9$ in polar form.

Solution. Replace x and y:

$$4(r\cos\theta)(r\sin\theta) = 9 \;\Rightarrow\; 4r^2\sin\theta\cos\theta = 9$$

Use $\sin 2\theta = 2\sin\theta\cos\theta$:

$$\boxed{2r^2\sin 2\theta = 9}$$

Part II

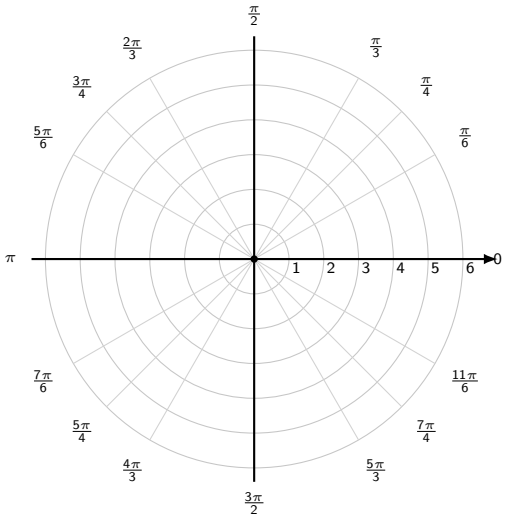
Polar Equations and Their Graphs

Grids Side by Side

A rectangular equation $y = f(x)$ collects points (x, y).

A polar equation such as $r = f(\theta)$ collects points (r, θ) that satisfy it.

Special attention: a point works if *any one* of its addresses satisfies the equation.



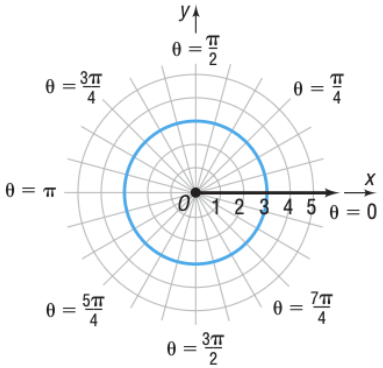
Example 1: $r = 3$

Identify and graph $r = 3$ by converting to rectangular form.

Solution.

$$r = 3 \;\Rightarrow\; r^2 = 9 \;\Rightarrow\; x^2 + y^2 = 9$$

A circle of radius 3 centered at the pole: every angle is allowed, the distance is always 3.



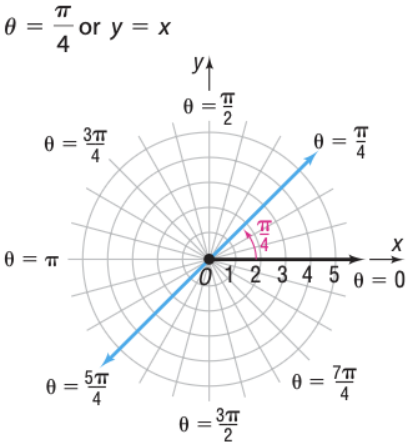
Example 2: $\theta = \frac{\pi}{4}$

Identify and graph $\theta = \frac{\pi}{4}$.

Solution.

$$\tan\theta = \frac{y}{x} \;\Rightarrow\; \tan\frac{\pi}{4} = \frac{y}{x} \;\Rightarrow\; 1 = \frac{y}{x} \;\Rightarrow\; y = x$$

A line through the pole with slope 1 (negative r covers the lower half).



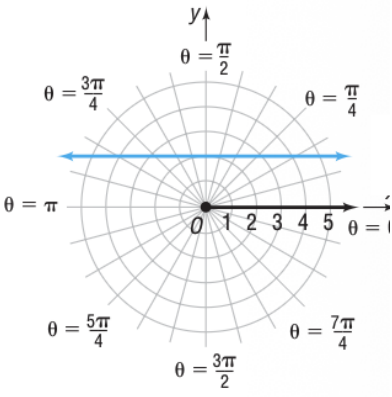
Example 3: $r \sin\theta = 2$

Convert $r \sin\theta = 2$ and describe its graph.

Solution. Since $y = r \sin\theta$:

$$\boxed{y = 2}$$

A horizontal line, 2 units above the polar axis.



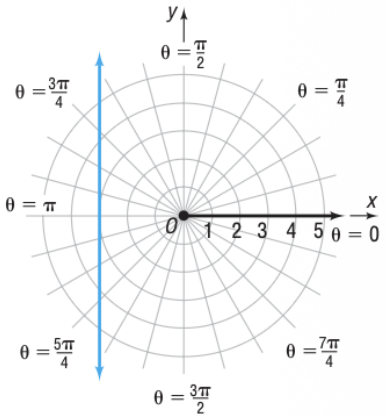
Example 4: $r\cos\theta = -3$

Convert $r\cos\theta = -3$ and describe its graph.

Solution. Since $x = r\cos\theta$:

$$\boxed{x = -3}$$

A vertical line, 3 units to the left of the pole.



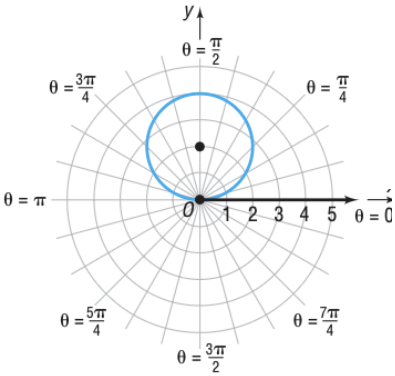
Example 5: $r = 4\sin\theta$

Identify and graph $r = 4\sin\theta$.

Solution. Multiply by r:

$$r^2 = 4r\sin\theta \Rightarrow x^2 + y^2 = 4y \Rightarrow x^2 + (y-2)^2 = 4$$

A circle: center $(0, 2)$, radius 2, sitting on the pole.



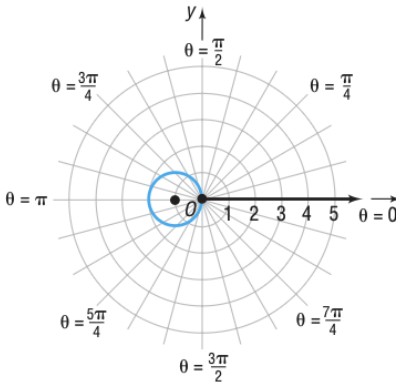
Example 6: $r = -2\cos\theta$

Identify and graph $r = -2\cos\theta$.

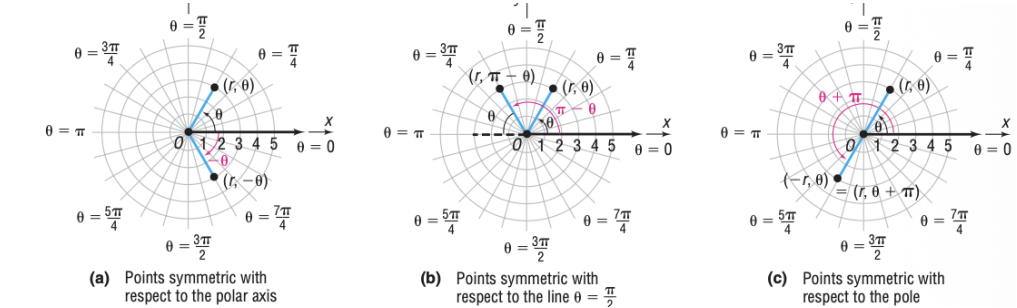
Solution. Multiply by r:

$$x^2 + y^2 = -2x \Rightarrow (x+1)^2 + y^2 = 1$$

A circle: center $(-1, 0)$, radius 1, through the pole.



Symmetry Tests: Work Smarter



(a) Points symmetric with respect to the polar axis

(b) Points symmetric with respect to the line $\theta = \frac{\pi}{2}$

(c) Points symmetric with respect to the pole

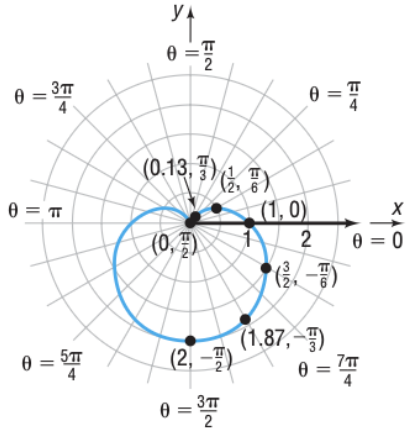
- **Polar axis** (the x-axis): replace θ by $-\theta$ and get the same equation.
- **Line** $\theta = \frac{\pi}{2}$ (the y-axis): replace θ by $\pi - \theta$.
- **Pole**: replace r by $-r$ (or θ by $\theta + \pi$).

Example 7: Cardioid $r = 1 - \sin\theta$

Graph $r = 1 - \sin\theta$.

Solution. Test $\theta \to \pi - \theta$:
$\sin(\pi - \theta) = \sin\theta$, so the graph is symmetric about the line $\theta = \frac{\pi}{2}$.
Plot $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ from the table, then reflect: a heart-shaped **cardioid** dipping to the pole at $\theta = \frac{\pi}{2}$.

θ	$r = 1 - \sin\theta$
$-\frac{\pi}{2}$	$1 - (-1) = 2$
$-\frac{\pi}{3}$	$1 - \left(-\frac{\sqrt{3}}{2}\right) \approx 1.87$
$-\frac{\pi}{6}$	$1 - \left(-\frac{1}{2}\right) = \frac{3}{2}$
0	$1 - 0 = 1$
$\frac{\pi}{6}$	$1 - \frac{1}{2} = \frac{1}{2}$
$\frac{\pi}{3}$	$1 - \frac{\sqrt{3}}{2} \approx 0.13$
$\frac{\pi}{2}$	$1 - 1 = 0$



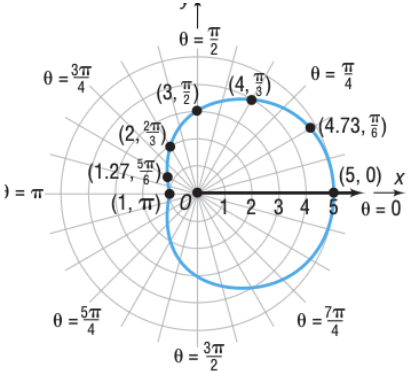
Example 8: Limaçon $r = 3 + 2\cos\theta$

Graph $r = 3 + 2\cos\theta$.

Solution. $\cos(-\theta) = \cos\theta$: symmetric about the polar axis, so plot $0 \le \theta \le \pi$ and reflect.

r never reaches 0 (its minimum is 1), so there is *no inner loop*: a **limaçon without a loop**.

θ	$r = 3 + 2\cos\theta$
0	$3 + 2(1) = 5$
$\frac{\pi}{6}$	$3 + 2\left(\frac{\sqrt{3}}{2}\right) \approx 4.73$
$\frac{\pi}{3}$	$3 + 2\left(\frac{1}{2}\right) = 4$
$\frac{\pi}{2}$	$3 + 2(0) = 3$
$\frac{2\pi}{3}$	$3 + 2\left(-\frac{1}{2}\right) = 2$
$\frac{5\pi}{6}$	$3 + 2\left(-\frac{\sqrt{3}}{2}\right) \approx 1.27$
π	$3 + 2(-1) = 1$

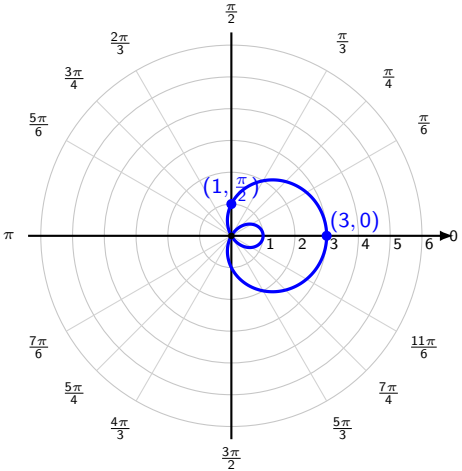


Example 9: Limaçon with a Loop, $r = 1 + 2\cos\theta$

Graph $r = 1 + 2\cos\theta$.

Solution. Symmetric about the polar axis. $r = 0$ when $\cos\theta = -\frac{1}{2}$, i.e. $\theta = \frac{2\pi}{3}$. After that $r < 0$, so those points are drawn on the *opposite* side: the **inner loop**.

θ	$r = 1 + 2\cos\theta$
0	$1 + 2(1) = 3$
$\frac{\pi}{6}$	$1 + 2\left(\frac{\sqrt{3}}{2}\right) \approx 2.73$
$\frac{\pi}{3}$	$1 + 2\left(\frac{1}{2}\right) = 2$
$\frac{\pi}{2}$	$1 + 2(0) = 1$
$\frac{2\pi}{3}$	$1 + 2\left(-\frac{1}{2}\right) = 0$
$\frac{5\pi}{6}$	$1 + 2\left(-\frac{\sqrt{3}}{2}\right) \approx -0.73$
π	$1 + 2(-1) = -1$

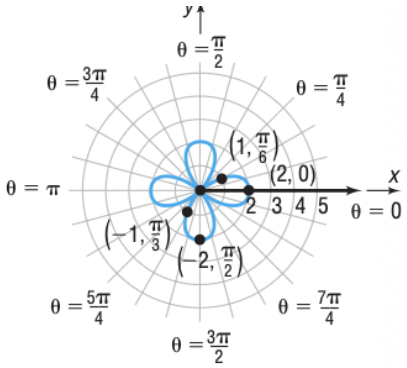


Example 10: Four-Petal Rose $r = 2\cos 2\theta$

Graph $r = 2\cos 2\theta$.

Solution. Replacing θ by $-\theta$, $\pi - \theta$ or r by $-r$ gives back the same equation, so the graph has all three symmetries. Plot $0 \le \theta \le \frac{\pi}{2}$, then reflect: a **rose** with four petals of length 2.

θ	$r = 2\cos(2\theta)$
0	$2(1) = 2$
$\frac{\pi}{6}$	$2\left(\frac{1}{2}\right) = 1$
$\frac{\pi}{4}$	$2(0) = 0$
$\frac{\pi}{3}$	$2\left(-\frac{1}{2}\right) = -1$
$\frac{\pi}{2}$	$2(-1) = -2$



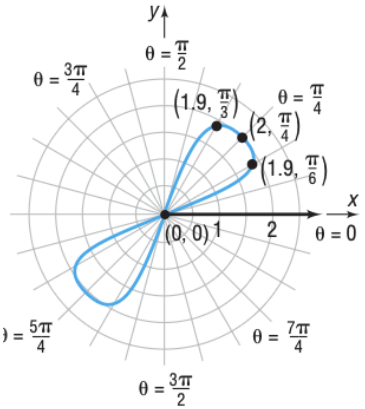
(b) $r = 2\cos(2\theta)$

Example 11: Lemniscate $r^2 = 4\sin 2\theta$

Graph $r^2 = 4\sin 2\theta$.

Solution. Symmetric about the pole ($r \to -r$ leaves r^2 alone). We need $\sin 2\theta \geq 0$, so θ lies in $[0, \frac{\pi}{2}]$ (and $[\pi, \frac{3\pi}{2}]$). Each θ gives $r = \pm 2\sqrt{\sin 2\theta}$: a figure-eight **lemniscate**.

θ	$r^2 = 4\sin(2\theta)$	r
0	$4(0) = 0$	0
$\frac{\pi}{6}$	$4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$	± 1.9
$\frac{\pi}{4}$	$4(1) = 4$	± 2
$\frac{\pi}{3}$	$4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$	± 1.9
$\frac{\pi}{2}$	$4(0) = 0$	0



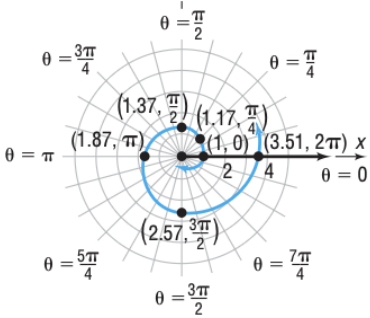
(b) $r^2 = 4\sin(2\theta)$

Example 12: Spiral $r = e^{\theta/5}$

Graph $r = e^{\theta/5}$.

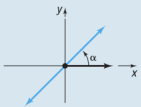
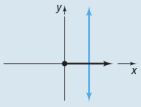
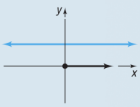
Solution. No symmetry holds. Since $e^{\theta/5} > 0$ and grows as θ grows, the curve spirals *outward* forever, and as $\theta \to -\infty$ it winds in toward (but never reaches) the pole: a **logarithmic spiral**.

θ	$r = e^{\theta/5}$
$-\frac{3\pi}{2}$	0.39
$-\pi$	0.53
$-\frac{\pi}{2}$	0.73
$-\frac{\pi}{4}$	0.85
0	1
$\frac{\pi}{4}$	1.17
$\frac{\pi}{2}$	1.37
π	1.87
$\frac{3\pi}{2}$	2.57
2π	3.51

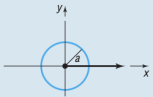
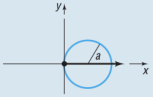
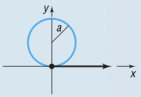


Meet the Curve Families

Lines

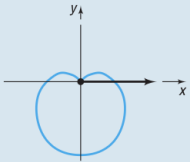
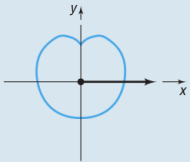
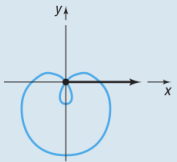
Description	Line passing through the pole making an angle α with the polar axis	Vertical line	Horizontal line
Rectangular equation	$y = (\tan \alpha)x$	$x = a$	$y = b$
Polar equation	$\theta = \alpha$	$r \cos \theta = a$	$r \sin \theta = b$
Typical graph			

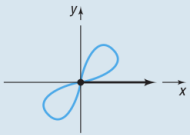
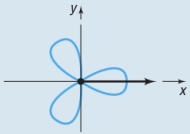
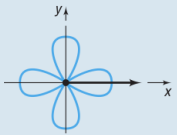
Circles

Description	Center at the pole, radius a	Passing through the pole, tangent to the line $\theta = \frac{\pi}{2}$, center on the polar axis, radius a	Passing through the pole, tangent to the polar axis, center on the line $\theta = \frac{\pi}{2}$, radius a
Rectangular equation	$x^2 + y^2 = a^2, \quad a > 0$	$x^2 + y^2 = \pm 2ax, \quad a > 0$	$x^2 + y^2 = \pm 2ay, \quad a > 0$
Polar equation	$r = a, \quad a > 0$	$r = \pm 2a \cos \theta, \quad a > 0$	$r = \pm 2a \sin \theta, \quad a > 0$
Typical graph			

Meet the Curve Families (continued)

Other Equations

Name	Cardioid	Limaçon without inner loop	Limaçon with inner loop
Polar equations	$r = a \pm a\cos\theta, \quad a > 0$	$r = a \pm b\cos\theta, \quad 0 < b < a$	$r = a \pm b\cos\theta, \quad 0 < a < b$
	$r = a \pm a\sin\theta, \quad a > 0$	$r = a \pm b\sin\theta, \quad 0 < b < a$	$r = a \pm b\sin\theta, \quad 0 < a < b$
Typical graph			

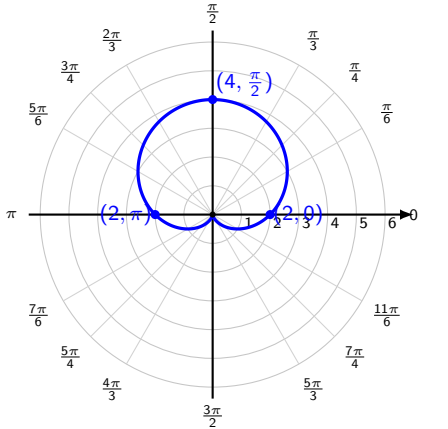
Name	Lemniscate	Rose with three petals	Rose with four petals
Polar equations	$r^2 = a^2\cos(2\theta), \quad a > 0$	$r = a\sin(3\theta), \quad a > 0$	$r = a\sin(2\theta), \quad a > 0$
	$r^2 = a^2\sin(2\theta), \quad a > 0$	$r = a\cos(3\theta), \quad a > 0$	$r = a\cos(2\theta), \quad a > 0$
Typical graph			

Example 13: Cardioid Check, $r = 2 + 2\sin\theta$

Graph $r = 2 + 2\sin\theta$.

Solution. Form $r = a + b\sin\theta$ with $a = b = 2$, so $\frac{a}{b} = 1$: a **cardioid**. Symmetric about $\theta = \frac{\pi}{2}$ ($\sin(\pi - \theta) = \sin\theta$). Values $r = 2, 4, 2, 0, 2$ at $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.

θ	$r = 2 + 2\sin\theta$
0	$2 + 2(0) = 2$
$\frac{\pi}{2}$	$2 + 2(1) = 4$
π	$2 + 2(0) = 2$
$\frac{3\pi}{2}$	$2 + 2(-1) = 0$
2π	$2 + 2(0) = 2$



Cheat Sheet

Convert points

$$x = r\cos\theta,\ y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2},\ \tan\theta = \frac{y}{x}$$

Check the quadrant for θ!

Convert equations

$$r^2 = x^2 + y^2$$

Multiply by r to expose x or y.

Graph

Test symmetry, tabulate, connect.