Lecture 10
The Law of Cosines and the Area of a Triangle

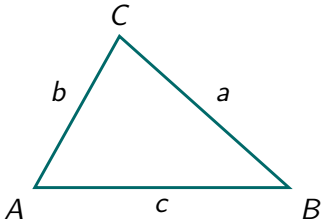
Pitambar Acharya

University of Alabama at Birmingham

October 26, 2026

Where We Are Headed

1. Crack the two cases the Law of Sines *cannot*: **SAS** and **SSS**
2. Use the Law of Cosines in a navigation problem
3. Measure the area of a triangle without an altitude
4. Meet Heron's remarkable formula



Part I

The Law of Cosines

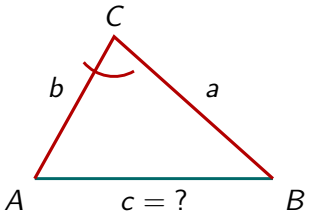
When the Law of Sines Runs Out of Road

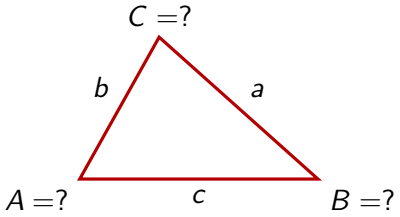
The Law of Sines needs a *matched pair*: an angle and the side opposite it. Two situations never give you one:

SAS two sides and the angle *between* them

SSS all three sides, no angle at all

We need a new tool — one that links sides *and* an angle in a single equation.





Theorem: The Law of Cosines

For a triangle with sides a, b, c opposite angles A, B, C:

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$a^2 = b^2 + c^2 - 2bc\cos A$$

In words: *the square of one side equals the sum of the squares of the other two, minus twice their product times the cosine of the angle between them.*

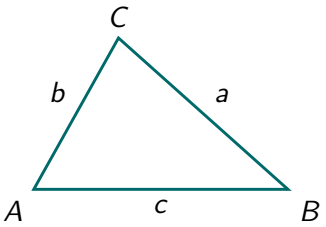
Note: Observe that if triangle is right angled, for instance $C = 90^\circ$ the last term vanishes — what famous theorem is hiding inside?

Putting the Law to Work

SAS: the angle you know sits between the two known sides, so the formula hands you the *third side* directly.

SSS: solve the formula for the cosine of an angle, then use $\cos^{-1}$:

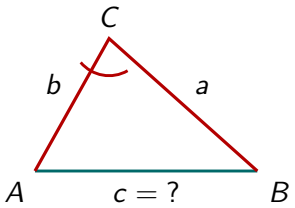
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$



Note: prefer the Law of Cosines for finding angles — unlike $\sin^{-1}$, the inverse cosine returns exactly one angle in $[0^\circ, 180^\circ]$, so there is no ambiguity.

Example: Two Sides and the Angle Between (SAS)

Solve the triangle with $a = 2$, $b = 3$, and $C = 60^\circ$.



$$c^2 = a^2 + b^2 - 2ab\cos C = 4 + 9 - 12\cos 60^\circ = 13 - 6 = 7 \implies c = \sqrt{7} \approx 2.65$$

Example: Finishing the SAS Triangle

Now that c is known, find angles A and B.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{9 + 7 - 4}{6\sqrt{7}} = \frac{2}{\sqrt{7}} \implies A = \cos^{-1}\left(\frac{2}{\sqrt{7}}\right) \approx 40.9^\circ$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{4 + 7 - 9}{4\sqrt{7}} = \frac{1}{2\sqrt{7}} \implies B \approx 79.1^\circ$$

Check: $60^\circ + 40.9^\circ + 79.1^\circ = 180^\circ$.

Example: Three Sides (SSS)

Solve the triangle with $a = 4$, $b = 3$, and $c = 6$.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{9 + 36 - 16}{36} = \frac{29}{36} \implies A \approx 36.3^\circ$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{16 + 36 - 9}{48} = \frac{43}{48} \implies B \approx 26.4^\circ$$

Example: Wrapping Up the SSS Triangle

Find the remaining angle using the two you already have, and sanity-check: do the three angles add to 180° and does the *largest* angle face the *longest* side?

$$C = 180^\circ - A - B \approx 180^\circ - 36.3^\circ - 26.4^\circ = 117.3^\circ$$

Check with the Law of Cosines:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{16 + 9 - 36}{24} = -\frac{11}{24} \implies C \approx 117.3^\circ \checkmark$$

The largest angle C faces the longest side $c = 6$.

Application

The Law of Cosines

Example - 1:

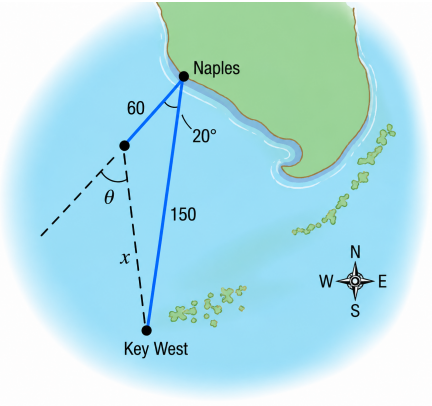
A motorized sailboat leaves Naples, Florida, bound for Key West, 150 miles away. It cruises at a steady 15 mph, but strong crosswinds and currents push it off course. After 4 hours it is 20° off the intended heading.

(a) How far is the boat from Key West now?

Distance travelled: $15 \cdot 4 = 60$ mi. Two sides and the included 20° angle are known:

$$x^2 = 150^2 + 60^2 - 2(150)(60)\cos 20^\circ \approx 9185.5$$

$$x \approx 95.8 \text{ miles}$$



Example - 1:

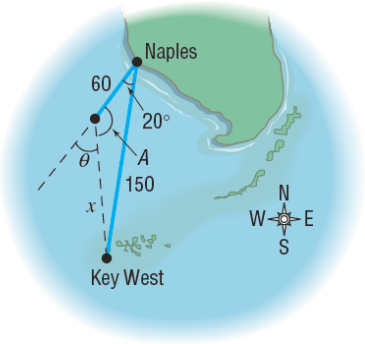
(b) Through what angle θ must the boat turn to head straight for Key West?

All three sides are now known. Find the interior angle A opposite the 150-mile side:

$$\cos A = \frac{60^2 + x^2 - 150^2}{2(60)(x)} \approx -0.845 \implies A \approx 147.6^\circ$$

θ and A are supplementary (straight line):

$$\theta = 180^\circ - A \approx 32.4^\circ$$



Example - 1:

(c) How much *extra* time does the detour add to the trip, if the boat stays at 15 mph?

New total path: $60 + 95.8 = 155.8$ miles, so the detour adds

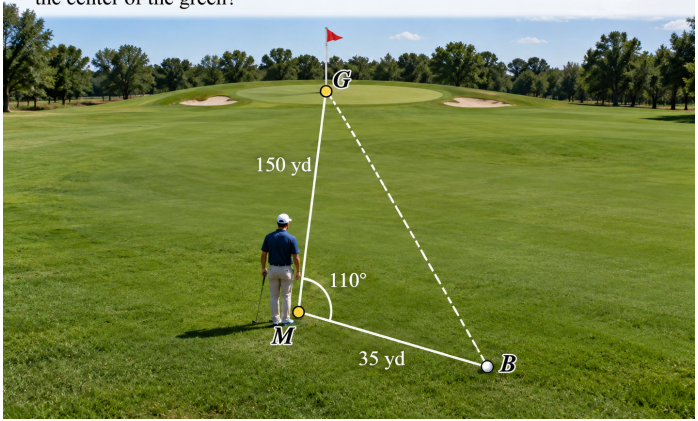
$$155.8 - 150 = 5.8 \text{ miles.}$$

Extra time:

$$\frac{5.8}{15} \approx 0.39 \text{ hour} \approx 23 \text{ minutes.}$$

Example - 2:

A golfer hits an errant tee shot that lands in the rough. A marker in the center of the fairway is 150 yards from the center of the green. While standing on the marker and facing the green, the golfer turns 110° toward his ball. He then paces off 35 yards to his ball. How far is the ball from the center of the green?



Example - 2: Solution

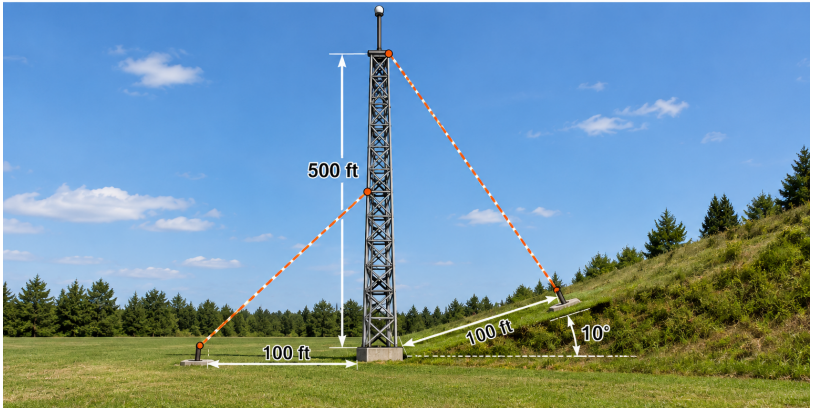
In triangle MGB two sides and the included 110° angle are known ($MG = 150$, $MB = 35$), so the Law of Cosines gives the distance $x = BG$:

$$\begin{aligned} x^2 &= 150^2 + 35^2 - 2(150)(35)\cos 110^\circ \\ &= 22500 + 1225 + 10500(0.3420) \approx 27{,}316 \\ x &\approx 165.3 \text{ yards} \end{aligned}$$

The ball is about 165 yards from the center of the green.

Example - 3:

The height of a radio tower is 500 feet, and the ground on one side of the tower slopes upward at an angle of 10° (see the figure). **(a)** How long should a guy wire be if it is to connect to the top of the tower and be secured at a point on the sloped side 100 feet from the base of the tower? **(b)** How long should a second guy wire be if it is to connect to the middle of the tower and be secured at a point 100 feet from the base on the flat side?



Example - 3: Solution

(a) The slope rises 10° above horizontal, so the angle between the vertical tower and the sloped ground is $90^\circ - 10^\circ = 80^\circ$. Two sides (500 and 100) and the included 80° angle:

$$\ell^2 = 500^2 + 100^2 - 2(500)(100)\cos 80^\circ \approx 242{,}635$$

$$\ell \approx 492.6 \text{ ft}$$

(b) The wire meets the tower 250 ft up (the middle), and the anchor is 100 ft out on level ground: the angle at the base is 90°.

$$\ell^2 = 250^2 + 100^2 - 0 = 72{,}500$$

$$\ell \approx 269.3 \text{ ft}$$

Law of Sines or Law of Cosines?

Given	First move	Why
ASA / SAA	Law of Sines	a matched pair is available
SSA	Law of Sines	watch for 0, 1, or 2 triangles
SAS	Law of Cosines	third side first
SSS	Law of Cosines	any angle first

Part II

The Area of a Triangle

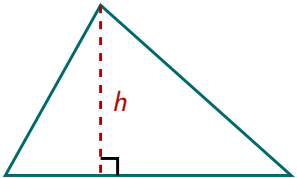
Starting Point: Base and Height

The area K of a triangle is

$$K = \frac{1}{2}\, b\, h$$

where b is a base and h the altitude drawn to it.

But what if the problem hands you *angles* and *sides*, and no altitude at all?



Challenge: No Altitude

In the triangle with sides a, b and included angle C, express the altitude h drawn to side b using a and $\sin C$. Then substitute into $K = \frac{1}{2}bh$.

In the right triangle formed by the altitude, a is the hypotenuse and h is the side opposite C:

$$\sin C = \frac{h}{a} \implies h = a \sin C$$

$$K = \tfrac{1}{2}\, b\, h = \tfrac{1}{2}\, ab \sin C$$

Theorem: Area from SAS

The area of a triangle is half the product of two sides times the sine of the angle between them:

$$K = \tfrac{1}{2}\, ab \sin C \;=\; \tfrac{1}{2}\, bc \sin A \;=\; \tfrac{1}{2}\, ac \sin B$$

Note: the angle must be the one *between* the two sides you pick. Mismatch them and the formula breaks.

Example: Area from Two Sides and an Angle

Find the area of the triangle with $a = 8$, $b = 6$, and $C = 30^\circ$.

C lies between sides a and b, so

$$K = \frac{1}{2}ab\sin C = \frac{1}{2}(8)(6)\sin 30^\circ = 24 \cdot \frac{1}{2} = 12 \text{ square units}$$

Theorem: Heron's Formula (SSS)

A triangle with sides a, b, c has area

$$K = \sqrt{s(s-a)(s-b)(s-c)}, \qquad s = \tfrac{1}{2}(a+b+c)$$

where s is the **semiperimeter**.

Example: Area from Three Sides

Find the area of a triangle whose sides measure 4, 5, and 7.

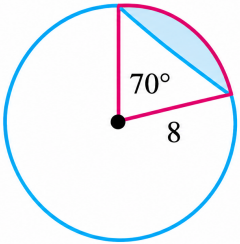
Note: simplify the radical: pull out the largest perfect-square factor.

$$s = \tfrac{1}{2}(4 + 5 + 7) = 8$$

$$K = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{8 \cdot 4 \cdot 3 \cdot 1} = \sqrt{96} = 4\sqrt{6} \approx 9.80 \text{ square units}$$

Additional Example - 1:

Area of a Segment. Find the area of the **segment** (shaded in blue in the figure) of a circle whose radius is 8 feet, formed by a central angle of 70°.



Additional Example - 1: Solution

Segment = sector − triangle, with $r = 8$ and central angle $70^\circ = \frac{7\pi}{18}$.

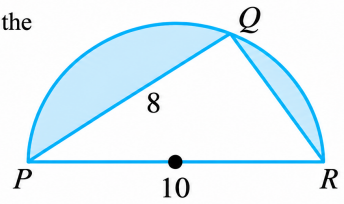
$$\text{Sector} = \frac{1}{2}r^2\theta = \frac{1}{2}(64)\frac{7\pi}{18} = \frac{112\pi}{9} \approx 39.10$$

$$\text{Triangle} = \frac{1}{2}r^2 \sin 70^\circ = 32 \sin 70^\circ \approx 30.07$$

$$\text{Segment} \approx 39.10 - 30.07 \approx 9.03 \text{ ft}^2$$

Additional Example - 2:

Computing Areas. See the figure. Find the area of the shaded region enclosed in a semicircle of diameter 10 inches. The length of the chord PQ is 8 inches.



Additional Example - 2: Solution

An angle inscribed in a semicircle is a right angle, so $\angle PQR = 90^\circ$ and $PR = 10$ is the hypotenuse.

$$QR = \sqrt{10^2 - 8^2} = 6, \qquad \text{Area}_{\triangle PQR} = \frac{1}{2}(8)(6) = 24$$

$$\text{Area}_{\text{semicircle}} = \frac{1}{2}\pi(5)^2 = 12.5\pi \approx 39.27$$

$$\text{Shaded} = 12.5\pi - 24 \approx 15.27 \text{ in}^2$$

Summary

- **Law of Cosines:** $c^2 = a^2 + b^2 - 2ab\cos C$ — the engine for SAS (find a side) and SSS (find an angle).
- **Choose wisely:** matched pair $\Rightarrow$ Law of Sines; otherwise Law of Cosines.
- **Area from SAS:** $K = \frac{1}{2}ab\sin C$.
- **Area from SSS:** Heron, $K = \sqrt{s(s-a)(s-b)(s-c)}$.