Lecture 9
Right-Triangle Applications and the Law of Sines

Objectives

1. Solve right triangles and apply them to real-world situations
2. Solve triangles when two angles and a side are known (ASA / SAA)
3. Navigate the ambiguous case (SSA): zero, one, or two triangles
4. Use the Law of Sines in applied problems

Part I

Applications Involving Right Triangles

Solving a Right Triangle

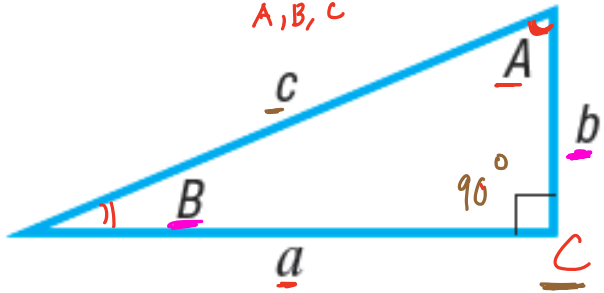
Lower case letter : sides
a, b, c
Upper case letter: Angles
A, B, C

To *solve* a right triangle means to find every missing side and angle. With the right angle at the corner opposite c:

$$a^2 + b^2 = c^2$$

are acute angles

$$A + B = 90^\circ$$



$$\sin A = \frac{a}{c}, \quad \cos A = \frac{b}{c}, \quad \tan A = \frac{a}{b}$$

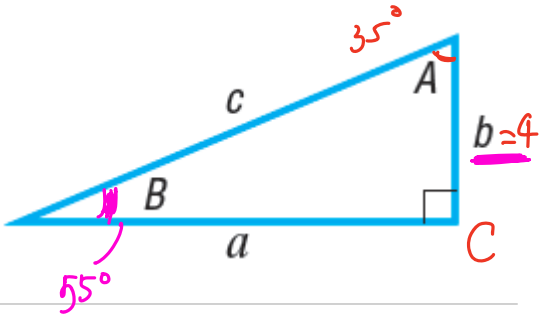
Special attention: set your calculator to *degree* mode before any inverse-trig step.

$A+B+C = 180^\circ$

Example: A Right Triangle from a Side and an Angle

In the triangle at right, $b = 4$ and $A = 35°$.
Find a, c, and B.

Special attention: the unknown angle comes free from $A + B = 90°$. For each side, ask which ratio links it to the side you already know.



$A + B = 90°$

$\Rightarrow 35° + B = 90°$

$\Rightarrow B = 90° - 35° = 55°$

$\tan(35°) = \frac{a}{b} = \frac{a}{4}$.

$\Rightarrow a = 4 \cdot \tan(35°)$

$a \approx 2.80$

$c = ?$

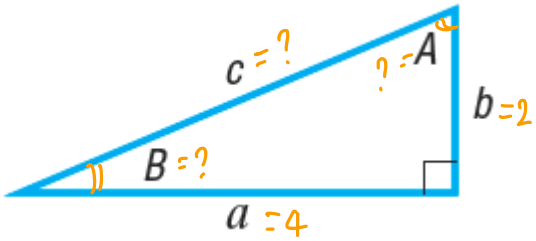
$\cos 35° = \frac{b}{c} = \frac{4}{c} \Rightarrow c = \frac{4}{\cos 35°}$

$c \approx 4.88$

Example: A Right Triangle from Two Sides

Now suppose $a = 4$ and $b = 2$.
Find c, A, and B (angles to one decimal place).

Special attention: get c from the Pythagorean Theorem; then use an *inverse* trig function to recover A.



For c:

$$c^2 = a^2 + b^2$$
$$= 4^2 + 2^2$$
$$= 16 + 4$$
$$= 20$$
$$\Rightarrow \boxed{c = \sqrt{20} = 2\sqrt{5} \approx 4.47}$$

What about A & B?

$$\tan A = \frac{a}{b} = \frac{4}{2} = 2 \Rightarrow \boxed{A = \tan^{-1}(2) = 63.4^\circ}$$

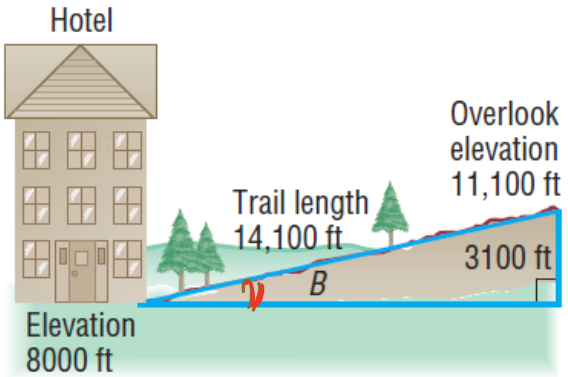
For B:

$$A + B = 90^\circ \Rightarrow \boxed{B = 90^\circ - 63.4^\circ = 26.6^\circ}$$

Example: How Steep Is the Mountain Trail?

A hotel sits at an elevation of 8000 ft. A straight trail leads to a scenic overlook at 11,100 ft, and the trail itself is 14,100 ft long. Find the trail's angle of inclination B.

Special attention: the *rise* is a difference of elevations. Which side of the right triangle is that, and which side is the trail?

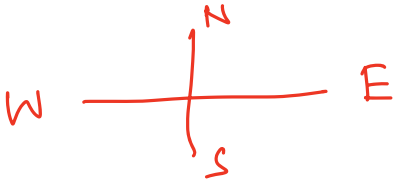


$$* \quad \sin B = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$= \frac{3100}{14100} \Rightarrow B = \sin^{-1}\left(\frac{3100}{14100}\right) \approx 12.7^\circ$$

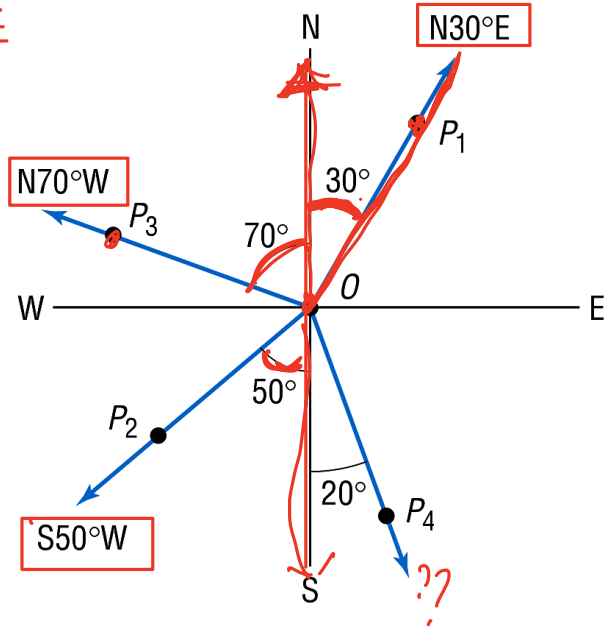
The inclination (grade) of the trail is approximately 12.7°.

Bearings: Describing Direction



In navigation and surveying, the **bearing** from a point O to a point P is the *acute* angle θ between ray OP and the north–south line through O.

It is written starting with N or S, then the angle, then E or W: for instance $\mathrm{N30^\circ E}$ or $\mathrm{S50^\circ W}$.

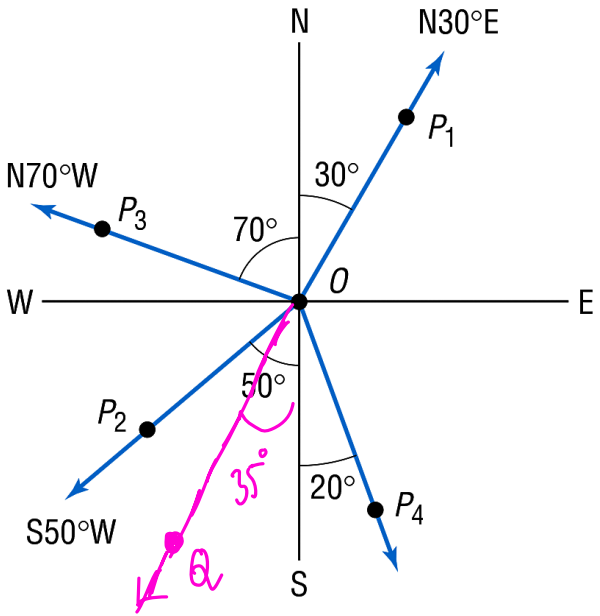


Example: Reading a Bearing

(a) Using the compass diagram, state the bearing from O to P_4.
(b) Sketch a point Q whose bearing from O is $\mathrm{S35^\circ W}$.

Special attention: always measure from the north–south line, never from the east–west line.

(a) S20°E



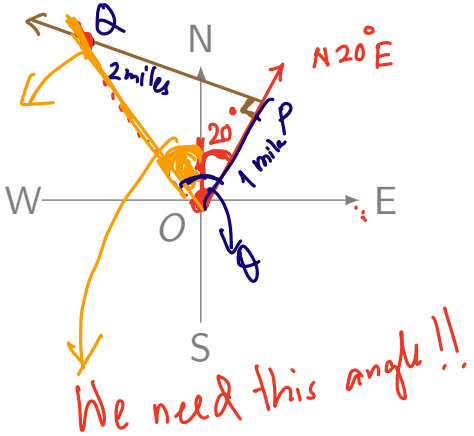
(b) S35°W

Example: Tracking an Airplane (1 of 2)

A Boeing 777 departs Nashville International on a runway with bearing $\mathrm{N}20^\circ\mathrm{E}$. After 1 mile, the pilot turns 90° toward the northwest and flies 2 more miles.

Question: what bearing should the control tower use to locate the aircraft?

Special attention: sketch first. Put the tower at the origin, mark the runway direction, then the 90° turn. The two flight legs are the legs of a right triangle.



Let θ be $\angle POQ$

Then:

$$\tan\theta = \frac{2}{1}$$

$$\Rightarrow \theta = \tan^{-1}(2)$$

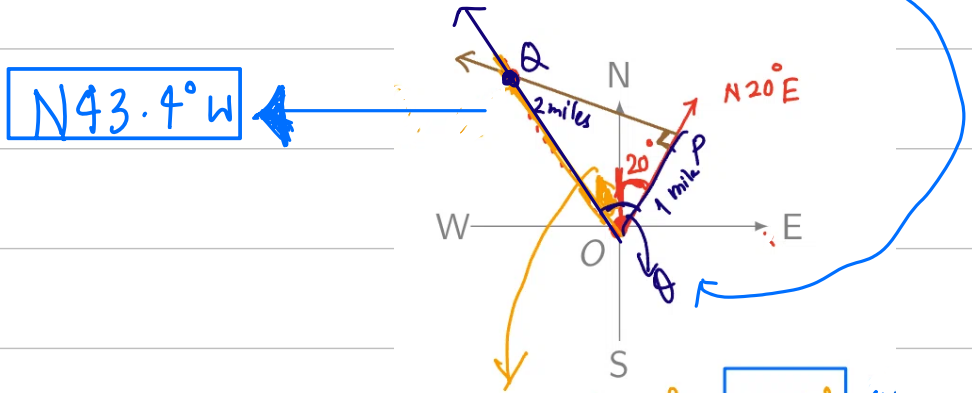
$$\approx 63.4^\circ$$

Let P, Q, O are the angles of right triangle OPQ.

Example: Tracking an Airplane (2 of 2)

Continue your sketch: let θ be the angle at the tower between the first flight leg and the line of sight to the aircraft.

Special attention: find θ with an inverse tangent, then combine it with the 20° runway angle. Be careful: the turn is toward the *west* of north.



N43.4° W

Right-Triangle Takeaways

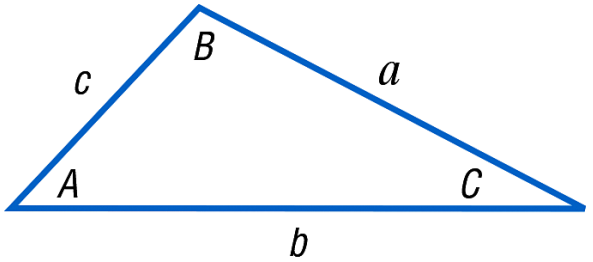
- Sketch and label before you compute.
- Pair the unknown with a known side through sin, cos, or tan.
- Complementary acute angles add to 90°.
- A bearing is always an *acute* angle measured from N or S.

Part II

The Law of Sines

Oblique Triangles

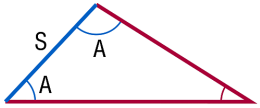
A triangle with no right angle is **oblique**. To solve one, we need one side plus any two further pieces of information: two angles, or one angle and another side, or the other two sides.



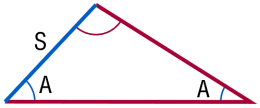
S + { AA → SAA/ASA
AS → SAS / SSA
SS

Four Starting Situations

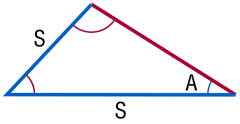
1. **ASA / SAA:** a side and two angles
2. **SSA:** two sides and an angle opposite one of them
3. **SAS:** two sides and the angle between them
4. **SSS:** all three sides



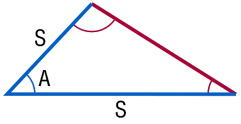
Case 1: ASA



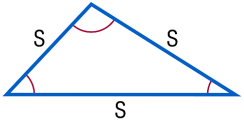
Case 1: SAA



Case 2: SSA



Case 3: SAS



Case 4: SSS

The Law of Sines handles cases 1 and 2.

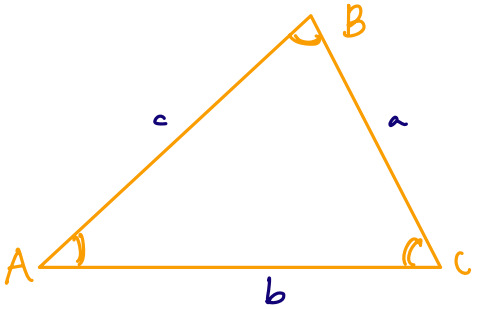
Theorem: The Law of Sines

For a triangle with sides a, b, c opposite angles A, B, C:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Also true

$$\boxed{\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}}$$



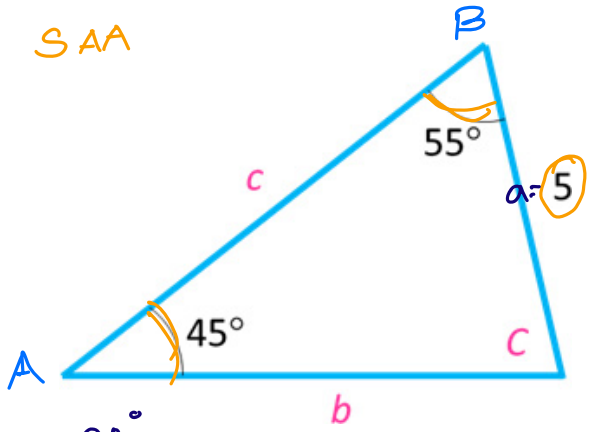
This packs three separate equalities:

$$\frac{\sin A}{a} = \frac{\sin B}{b} \qquad \frac{\sin A}{a} = \frac{\sin C}{c} \qquad \frac{\sin B}{b} = \frac{\sin C}{c}$$



Special attention: also keep $A + B + C = 180^\circ$ close at hand. Each application of the Law uses *one matched pair* — an angle and the side opposite it — that you already know.

Example: Two Angles and a Side (SAA)



Solve the triangle with $A = 45°$, $B = 55°$, and $a = 5$.

Special attention: find C first. Then apply the Law of Sines twice — once for b, once for c — each time pairing with the known pair A and a.

$$A + B + C = 180°$$

$$C = 180° - A - B = 180° - 45° - 55° = 80°$$

$a = 5$ given; find b and c. → we apply sine law:

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\Rightarrow b = \frac{a \cdot \sin B}{\sin A} = \frac{5 \cdot \sin 55°}{\sin 45°} \approx 5.79$$

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

$$\Rightarrow c = \frac{a \sin C}{\sin A} = \frac{5 \cdot \sin 80°}{\sin 45°} \approx 6.96$$

Example: Two Angles and the Included Side (ASA)

Solve the triangle with $A = 20°$, $B = 25°$, and $c = 4$.

Special attention: the known side c is opposite C, so first compute C to create the matched pair (C, c). Then solve for a and b.

$C = 180° - A - B$ ($A+B+C = 180°$)

$= 180° - 20° - 25°$

$= 135°$

$c = 4$ given. Need to find a & b.

Apply Sine Law:

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

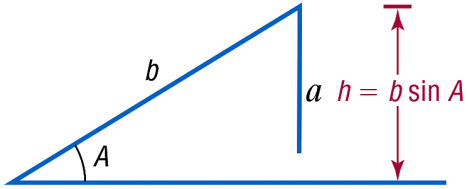
$$\Rightarrow a = c \cdot \frac{\sin A}{\sin C} = \frac{4 \cdot \sin 20°}{\sin 135°} \approx 1.93$$

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

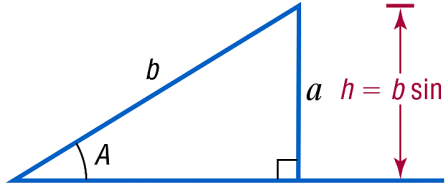
$$\Rightarrow b = \frac{c \sin B}{\sin C} \approx 2.39.$$

The Ambiguous Case (SSA)

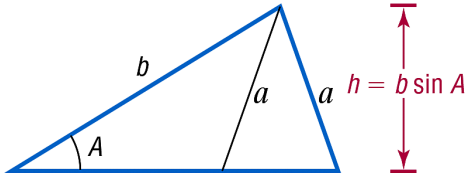
With two sides and an angle *opposite one of them*, the data may produce **no triangle, one, or two**. Let $h = b \sin A$ be the height measured from the end of side b down to the base.



$a < h = b \sin A$



$a = h = b \sin A$

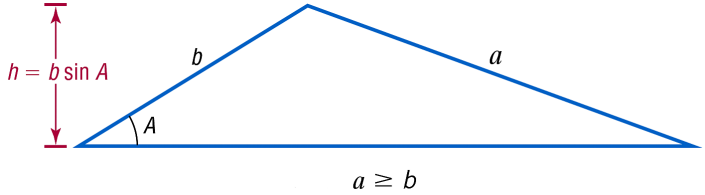


$b \sin A < a$ and $a < b$

Sorting the SSA Outcomes

Compare side a with $h = b\sin A$ and with b:

- $a < h \quad \Rightarrow \quad$ no triangle
- $a = h \quad \Rightarrow \quad$ one (right) triangle
- $h < a < b \quad \Rightarrow \quad$ two triangles
- $a \geq b \quad \Rightarrow \quad$ one triangle



Special attention: in practice, let the Law of Sines decide: if sin(angle) > 1 there is no triangle; otherwise test both the acute angle and its supplement $180° -$(angle).

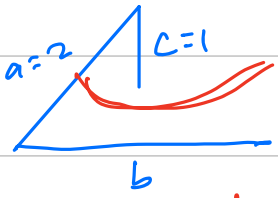
Example: SSA — Does a Triangle Exist?

Solve the triangle with $a = 2$, $c = 1$, and $C = 50^\circ$.

Special attention: use $\frac{\sin A}{a} = \frac{\sin C}{c}$ to get $\sin A$. Then ask: can a sine ever take that value?

Solution:

$$2 \cdot \frac{\sin A}{2} = \frac{\sin 50^\circ}{1} \cdot 2$$



$$\Rightarrow \sin A = 2 \cdot \sin 50^\circ$$

$$\approx 1.532 \quad \text{(Impossible)} \Rightarrow$$

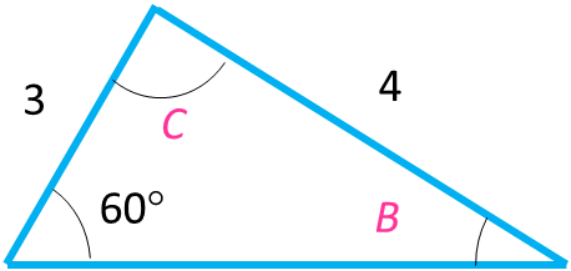
No triangle exists!!

We know: the value of sine function lies in $[-1, 1]$

Example: SSA — A Single Triangle (1 of 2)

Solve the triangle with $a = 4$, $b = 3$, and $A = 60°$.

Special attention: find $\sin B$ first. There will be two candidate angles for B between $0°$ and $180°$ — test each against $A + B < 180°$.



$a = 4, \quad b = 3, \quad A = 60°$

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\Rightarrow \sin B = \frac{b \cdot \sin A}{a}$$

$$\sin B = \frac{3 \cdot \sin 60°}{4} = \frac{3}{4} \cdot \frac{\sqrt{3}}{2} \approx 0.65 > 0$$

$$\Rightarrow B_1 = \sin^{-1}(0.65) = 40.5°$$

Since $\sin B > 0$ there are two angles in I & II quadrant for which $\sin B$ is +ve.



$B_2 = 180 - 40.5° = 139.5°$ (INVALID)

Since, $A + B_2 > 180°$

Example: SSA — A Single Triangle (2 of 2)

Finish the problem above: once the valid B survives, find C and then side c.

Special attention: $C = 180^\circ - A - B$, then apply the Law of Sines once more with the known pair (A, a).

Since, B_1 is valid as $A + B_1 < 180^\circ$

Then, to determine C:

$A + B_1 + C = 180^\circ \Rightarrow C = 180^\circ - 40.5^\circ - 60^\circ$

$= 79.5^\circ$

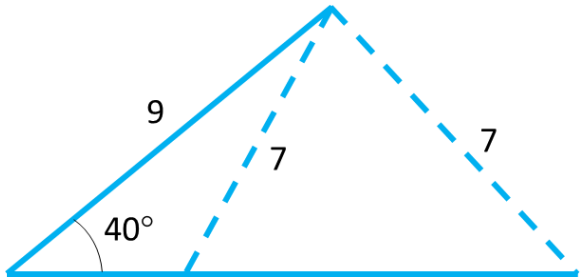
Find c: $\frac{\sin C}{c} = \frac{\sin A}{a}$

$\Rightarrow c = \frac{a \cdot \sin C}{\sin A} = \frac{4 \cdot \sin(79.5^\circ)}{\sin 60^\circ} = \ldots$ (do yourself)

Example: SSA — Two Triangles (1 of 2)

Solve the triangle with $a = 7$, $b = 9$, and $A = 40°$.

Special attention: compute $\sin B$ and record *both* angles B_1 (acute) and $B_2 = 180° - B_1$. Check that each satisfies $A + B < 180°$.



$a = 7$, $b = 9$, $A = 40^\circ$

To determine B: $\frac{\sin B}{b} = \frac{\sin A}{a}$

$\Rightarrow \sin B = \frac{b \cdot \sin A}{a} = \frac{9 \cdot \sin 40^\circ}{7} \Rightarrow B_1 = \sin^{-1}\left(\frac{9 \cdot \sin 40^\circ}{7}\right) \approx 55.7^\circ$

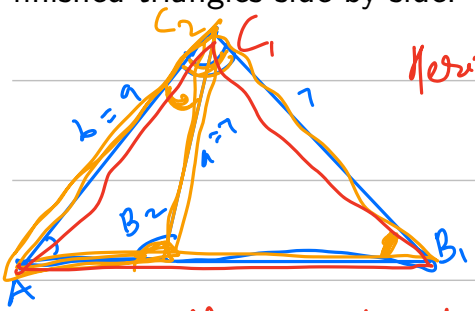
$B_2 = 180^\circ - 55.7^\circ = 124.3^\circ$

So, B_1 & B_2 are admissible. as $A + B_1 < 180^\circ$, $A + B_2 < 180^\circ$

Example: SSA — Two Triangles (2 of 2)

For each of the two triangles found above, compute the third angle C_i and the third side c_i.

Special attention: do the whole computation twice, once with B_1 and once with B_2. Present the two finished triangles side by side.



Here, $A = 40^\circ$ $\quad (A+B+C=180^\circ)$

$B_1 = 55.7^\circ$

$C_1 = 180^\circ - A - B_1 = 180^\circ - 40^\circ - 55.7^\circ \approx 84.3^\circ$

Also, $A = 40^\circ$

$B_2 = 124.3^\circ$

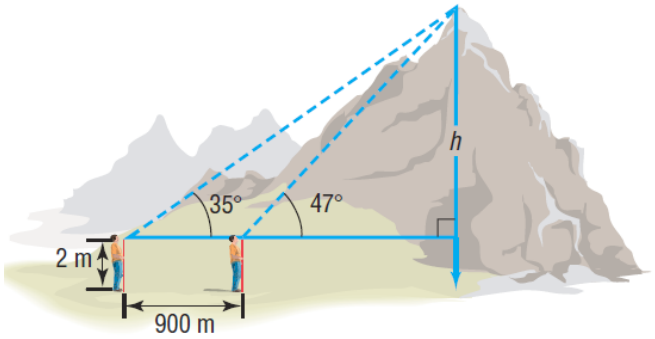
$C_2 = 180^\circ - 40^\circ - 124.3^\circ = 15.7^\circ$

So, we get two triangles: $\boxed{AB_1C_1}$ and $\boxed{AB_2C_2}$

Example: Height of a Mountain (1 of 2)

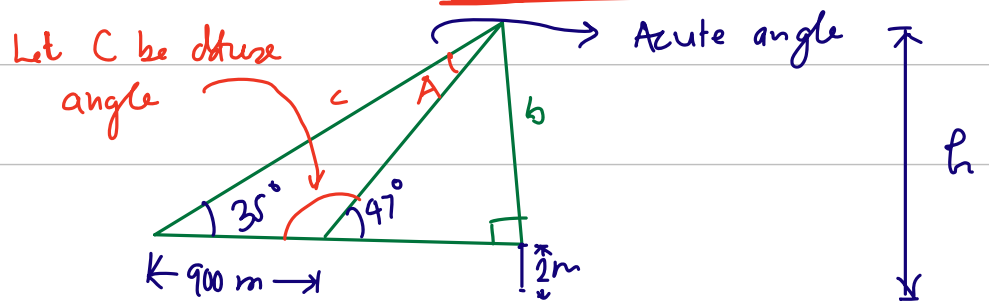
A surveyor sights the peak twice along a straight line toward the mountain, from two spots 900 m apart. The angles of elevation are 35° (farther spot) and 47° (nearer spot). The transit stands 2 m above the ground.

Find the height h of the mountain.



Special attention: sketch the two sight-lines to find an *oblique* triangle with the 900 m baseline.

SKETCH:



Example: Height of a Mountain (2 of 2)

Continue: use the angles of elevation to find the angles of the oblique triangle, apply the Law of Sines to get the sight-line to the peak, and then a right triangle to get the height. Do not forget the transit height.

Special attention: an angle of elevation of 47° gives an exterior angle at the nearer spot. Use 180° sums to find the triangle's angles.

since $C + 47^\circ = 180^\circ \Rightarrow C = 133^\circ$

Also, since $A + C + 35^\circ = 180^\circ \Rightarrow A = 180^\circ - 35^\circ - 133^\circ = 12^\circ$

We use Sine law to find c: $\frac{\sin A}{a} = \frac{\sin C}{c} \Rightarrow c = \frac{a \sin C}{\sin A} = \frac{900 \cdot \sin 133^\circ}{\sin 12^\circ} = 3165.86$

Using Larger right triangle:

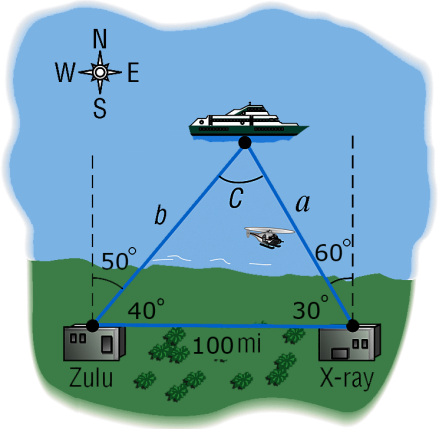
$\sin 35^\circ = \frac{b}{c} \Rightarrow b = c \cdot \sin 35^\circ \approx 1816$ m

So height of mountain $= b + 2 = 1816 + 2 = 1818$ m.

Example: Rescue at Sea (1 of 2)

Station Zulu is 100 miles due west of Station X-ray. A ship sends an SOS received by both stations: the ship's bearing is $\mathrm{N}50^\circ\mathrm{E}$ from Zulu and $\mathrm{N}60^\circ\mathrm{W}$ from X-ray.

(a) How far is each station from the ship?



Special attention: turn the bearings into interior angles of the triangle (Zulu, X-ray, ship); then find the angle at the ship.

→ figure above illustrates the situation. The angle C is:

$40^\circ + 30^\circ + C = 180^\circ \Rightarrow C = 180^\circ - 70^\circ$

$= 110^\circ$

Example: Rescue at Sea (2 of 2)

Finish the rescue: apply the Law of Sines for the two distances.

(b) A helicopter flying 200 mph launches from the *nearer* station. How long does it take to reach the ship?

Special attention: use distance = rate × time, and convert hours to minutes at the end.

(a) Using Law of Sines: $\frac{\sin 40^\circ}{a} = \frac{\sin 110^\circ}{100}$

$\Rightarrow a = (100)\frac{\sin 40^\circ}{\sin 110^\circ}$

$a \approx 68.40$ miles

Also: $\frac{\sin 30^\circ}{b} = \frac{\sin 110^\circ}{100}$

$\Rightarrow b = \frac{(100)\sin 30^\circ}{\sin 110^\circ}$

$b \approx 53.21$ miles

(b) The time t needed for the helicopter to reach the ship from station Zulu is:

(Rate, r)(Time, t) = Distance, a

$\Rightarrow t = \frac{b}{r} = \frac{53.21}{200} \approx 0.27$ hour

$\approx$ 16 minutes

It will take about 16 minutes to reach the ship.

Summary

- **Right triangles:** Pythagoras, sin / cos / tan, and $A + B = 90^\circ$ solve everything; bearings are acute angles from N or S.
- **Law of Sines:** pair each angle with its opposite side, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.
- **ASA / SAA:** get the third angle, then apply the Law twice.
- **SSA:** test for sin > 1, and for two supplementary angles — never assume one answer.