Lecture 8
Trigonometric Identities and Sum, Double-Angle, and Half-Angle Formulas

October 12, 2026

Objectives

1. Use Algebra to Establish Trigonometric Identities
2. Use Sum and Difference Formulas to Find Exact Values and Establish Identities
3. Use Double-Angle and Half-Angle Formulas to Find Exact Values and Establish Identities

Identity vs. Conditional Equation

$\sin^2 x + \cos^2 x = 1$

True for: $x = \frac{\pi}{3}, \frac{\pi}{6}, \frac{\pi}{2}, \pi, \ldots$

Two functions f and g are called **identically equal** if

$$f(x) = g(x)$$

for *every* value of x for which both sides are defined. Such an equation is an **identity**.

An equation that holds for only *some* values of x - not all - is a **conditional equation**.

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Special attention: proving an identity is not like solving an equation. You never move terms across the equals sign; instead you rewrite one side, step by step, until it matches the other side exactly.

Quotient and Reciprocal Identities

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Pythagorean and Even–Odd Identities

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad \cot^2\theta + 1 = \csc^2\theta$$

Even–Odd Identities

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$

$$\csc(-\theta) = -\csc\theta \qquad \sec(-\theta) = \sec\theta \qquad \cot(-\theta) = -\cot\theta$$

Objective 1

Use Algebra to Simplify Trigonometric Expressions

Four Go-To Algebraic Moves

Establishing an identity almost always comes down to one (or more) of these moves:

1. Rewrite every function in terms of sine and cosine only.
2. Multiply the numerator and denominator of a fraction by a "well-chosen 1."
3. Combine a sum or difference of ratios into a single fraction.
4. Factor, then cancel common pieces.

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3. Combine a sum or difference of ratios into a single fraction.
4. Factor, then cancel common pieces.

Special attention: these are tools, not a fixed order : look at the expression first and decide which move actually helps.

Example: Simplifying with Algebra

Work out each of the following, showing every algebraic step:

(a) Simplify $\dfrac{\tan\theta}{\sec\theta}$ by rewriting both functions in terms of sine and cosine.

(b) Show that $\dfrac{\sin\theta}{1-\cos\theta} = \dfrac{1+\cos\theta}{\sin\theta}$ by multiplying numerator and denominator by $1+\cos\theta$.

(c) Simplify $\dfrac{\cos^2\theta - 1}{\tan\theta\cos\theta + \tan\theta}$ by factoring.

Solution:

a) $\dfrac{\tan\theta}{\sec\theta} = \tan\theta \cdot \dfrac{1}{\sec\theta} = \dfrac{\sin\theta}{\cos\theta} \cdot \cos\theta = \sin\theta$ ☒

b) $L\cdot H\cdot S = \dfrac{\sin\theta}{1-\cos\theta}$

$$\frac{\sin\theta}{1-\cos\theta} = \frac{\sin\theta}{1-\cos\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta} = \frac{\sin\theta(1+\cos\theta)}{(1-\cos\theta)(1+\cos\theta)} = \frac{\sin\theta(1+\cos\theta)}{1-\cos^2\theta}$$

$$= \frac{\sin\theta(1+\cos\theta)}{\sin^2\theta} = \frac{\cancel{\sin\theta}\cdot(1+\cos\theta)}{\cancel{\sin\theta}\cdot\sin\theta}$$

$$= \frac{1+\cos\theta}{\sin\theta} \rightarrow \text{RHS}$$

$A^2 - B^2 = (A+B)(A-B)$

$\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow 1-\cos^2\theta = \sin^2\theta$

$$\frac{\cos^2\theta - 1^2}{\tan\theta\cdot\cos\theta + \tan\theta} = \frac{(\cos\theta+1)(\cos\theta-1)}{\tan\theta(\cos\theta+1)}$$

$$= \frac{\cos\theta - 1}{\tan\theta}$$

Objective 2

Establish Trigonometric Identities

Example: Establishing an Identity

Establish the identity:

$$\csc\theta \cdot \tan\theta = \sec\theta$$

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Establish the identity:

$$\csc\theta \cdot \tan\theta = \sec\theta$$

Special attention: start from the side that looks more complicated. Here, that's the left, and rewrite until it turns into the right side.

Left side $= \csc\theta \cdot \tan\theta$

$= \dfrac{1}{\sin\theta} \cdot \dfrac{\sin\theta}{\cos\theta}$

$= \dfrac{1}{\cos\theta}$

$= \sec\theta \rightarrow$ Right side.

Example: Establishing an Identity

Establish the identity:

$$\sin^2(-\theta) + \cos^2(-\theta) = 1$$

Example: Establishing an Identity

Establish the identity:

$$\sin^2(-\theta) + \cos^2(-\theta) = 1$$

Special attention: the arguments are $-\theta$, so start by applying the Even–Odd Identities before anything else.

$$\sin^2(-\theta) + \cos^2(-\theta) = [\sin(-\theta)]^2 + [\cos(-\theta)]^2$$

$$= (-\sin\theta)^2 + (\cos\theta)^2$$

$$= \sin^2\theta + \cos^2\theta$$

$$= 1$$

Example: Establishing an Identity

Establish the identity:

$$\frac{\sin^2(-\theta) - \cos^2(-\theta)}{\sin(-\theta) - \cos(-\theta)} = \cos\theta - \sin\theta$$

Example: Establishing an Identity

$A^2 - B^2 = (A+B)(A-B)$

Establish the identity:

$$\frac{\sin^2(-\theta) - \cos^2(-\theta)}{\sin(-\theta) - \cos(-\theta)} = \cos\theta - \sin\theta$$

Special attention: again the arguments are $-\theta$ on the left but θ on the right. So, convert first, then look for a difference of squares to factor.

Left hand:

$$\frac{\sin^2(-\theta) - \cos^2(-\theta)}{\sin(-\theta) - \cos(-\theta)} = \frac{[\sin(-\theta)]^2 - [\cos(-\theta)]^2}{-\sin\theta - \cos\theta}$$

[$\sin\theta$
$\sin(-\theta) = -\sin\theta$
$\cos(-\theta) = \cos\theta$]

$$= \frac{(-\sin\theta)^2 - \cos^2\theta}{-\sin\theta - \cos\theta} = \frac{\sin^2\theta - \cos^2\theta}{-(\sin\theta + \cos\theta)}$$

$$= \frac{(\sin\theta + \cos\theta)(\sin\theta - \cos\theta)}{-(\sin\theta + \cos\theta)} = -(\sin\theta - \cos\theta)$$

$= -\sin\theta + \cos\theta$

$\cos\theta - \sin\theta$
Right hand

Example: Establishing an Identity

Establish the identity:

$$\frac{1+\sin\theta}{1+\csc\theta} = \sin\theta$$

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Establish the identity:

$$\frac{1+\sin\theta}{1+\csc\theta} = \sin\theta$$

Special attention: rewrite $\csc\theta$ as $\dfrac{1}{\sin\theta}$ first, then combine the denominator into a single fraction before simplifying.

Left side: $\dfrac{1+\sin\theta}{1+\csc\theta} = \dfrac{1+\sin\theta}{1+\left(\frac{1}{\sin\theta}\right)} = \dfrac{1+\sin\theta}{\left(\frac{\sin\theta+1}{\sin\theta}\right)} = (1+\sin\theta)\cdot\dfrac{\sin\theta}{(\sin\theta+1)} = \sin\theta$

$1+\dfrac{1}{\sin\theta}$

$= \dfrac{\sin\theta}{\sin\theta} + \dfrac{1}{\sin\theta} = \dfrac{(\sin\theta+1)}{\sin\theta}$

Example: Establishing an Identity

Establish the identity:

$$\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} = 2\csc\theta$$

$a^2 + 2ab + b^2 = (a+b)^2$

Special attention: combine the left side over a common denominator first: a Pythagorean Identity will appear in the numerator once you expand.

$$\rightarrow \frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} = \left(\frac{\sin\theta}{1+\cos\theta}\right)\cdot\frac{\sin\theta}{\sin\theta} + \frac{(1+\cos\theta)}{\sin\theta}\cdot\frac{(1+\cos\theta)}{(1+\cos\theta)}$$

$$= \frac{\sin^2\theta + (1+\cos\theta)^2}{(\sin\theta)(1+\cos\theta)}$$

$1 + 2\cos\theta\cdot 1 + \cos^2\theta = 1 + 2\cos\theta + \cos^2\theta$

$$= \frac{\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta}{(\sin\theta)(1+\cos\theta)} = \frac{1+1+2\cos\theta}{\sin\theta(1+\cos\theta)}$$

$$= \frac{2+2\cos\theta}{\sin\theta(1+\cos\theta)} = \frac{2(1+\cos\theta)}{\sin\theta(1+\cos\theta)} = \frac{2}{\sin\theta} = 2\csc\theta$$

Example: Establishing an Identity

Establish the identity:

$$\frac{\tan v + \cot v}{\sec v \csc v} = 1$$

Special attention: rewrite the numerator and denominator entirely in sine and cosine: the whole expression collapses quickly once you do.

$\hookrightarrow$ LHS:

$$\frac{\frac{\sin v}{\cos v} + \frac{\cos v}{\sin v}}{\sec v \cdot \csc v} = \frac{\left(\frac{\sin^2 v + \cos^2 v}{\cos v \cdot \sin v}\right)}{\sec v \cdot \csc v} \qquad (\sin^2 v + \cos^2 v = 1)$$

$$= \frac{1}{\cos v \cdot \sin v} \cdot \frac{1}{\sec v} \cdot \frac{1}{\csc v} = \frac{1}{\cos v \cdot \sin v} \cdot \cos v \cdot \sin v = 1$$

Example: Establishing an Identity

Establish the identity:

$$\frac{1-\sin\theta}{\cos\theta} = \frac{\cos\theta}{1+\sin\theta}$$

$A^2 - B^2 = (A+B)(A-B)$

Special attention: multiply the left side by $\frac{1+\sin\theta}{1+\sin\theta}$; a Pythagorean Identity will clear the $\sin^2\theta$ term in the numerator.

Left side := $\frac{1-\sin\theta}{\cos\theta} \cdot 1 = \frac{(1-\sin\theta)}{\cos\theta} \cdot \frac{(1+\sin\theta)}{1+\sin\theta} = \frac{(1-\sin\theta)(1+\sin\theta)}{\cos\theta(1+\sin\theta)}$ $\quad (A-B)(A+B)$

$= \frac{1-\sin^2\theta}{\cos\theta(1+\sin\theta)} = \frac{\cos^2\theta}{\cos\theta(1+\sin\theta)}$ $\quad [\cos^2\theta = 1-\sin^2\theta]$

$= \frac{\cos\theta}{1+\sin\theta} \rightarrow$ Right side.

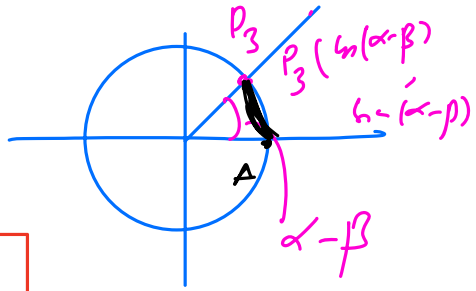
Guidelines for Establishing Identities

1. Start with the side that looks more complicated.
2. Rewrite a sum or difference of ratios as a single ratio.
3. When stuck, rewrite everything in terms of sine and cosine.
4. Keep the target side in view — let its shape guide which move to try next.

Objective 3

Use Sum and Difference Formulas to Find Exact Values

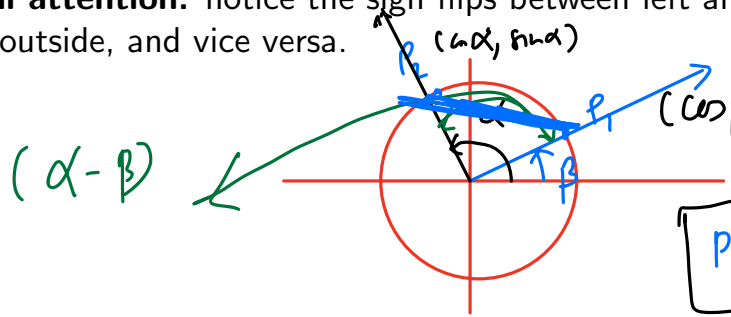
Theorem: Sum and Difference Formulas for Cosine



$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Special attention: notice the sign flips between left and right expressions - a *plus* inside becomes a *minus* outside, and vice versa.



$P_1P_2 = P_3A$ $\Rightarrow$ $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Example: Using the Sum Formula for an Exact Value

Find the exact value of $\cos 105^\circ$.

Example: Using the Sum Formula for an Exact Value

Find the exact value of $\cos 105°$.

Note: $105°$ is not a standard angle, but it *is* a sum of two standard angles you already know the sine and cosine of. Which two?

$$105° = (60°) + (45°)$$

$$\cos(105°) = \cos(\underbrace{60°}_{\alpha} + \underbrace{45°}_{\beta}) = \cos 60° \cdot \cos 45° - \sin 60° \cdot \sin 45°$$

$$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{2} - \sqrt{6}}{4}$$

$\cos 75°$

$$\frac{\sqrt{6} - \sqrt{2}}{4}$$

Example: Using the Difference Formula for an Exact Value

Find the exact value of $\cos \frac{5\pi}{12}$.

Example: Using the Difference Formula for an Exact Value

Find the exact value of $\cos\frac{5\pi}{12}$.

Note: write $\frac{5\pi}{12}$ as a sum (or difference) of two angles from $\left\{\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}\right\}$ with a common denominator of 12.

$$\cos\left(\frac{5\pi}{12}\right) = \cos\left(\frac{3\pi}{12} - \frac{2\pi}{12}\right) = \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$\frac{\pi}{4}$ (α), $\frac{\pi}{6}$ (β)

You should LEARN how to break the given angle into two standard angles.

$$= \cos\frac{\pi}{4}\cdot\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6}$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}$$

$$= \frac{\sqrt{6}+\sqrt{2}}{4}.$$

Co-function Identities

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta \qquad \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

Try deriving them..

$\cos\frac{\pi}{2}\cdot\cos\theta + \sin\frac{\pi}{2}\cdot\sin\theta$

$\cos\frac{\pi}{2} = 0$, $\sin\frac{\pi}{2} = 1$

$= 0 + \sin\theta$

$= \sin\theta$

Theorem: Sum and Difference Formulas for Sine

If $\alpha = \beta = \theta$

$\sin(2\theta) = \sin(\alpha + \beta) = \sin\theta \cdot \cos\theta + \cos\theta \cdot \sin\theta$

$= 2\sin\theta \cdot \cos\theta$

$\sin 2\theta = 2\sin\theta \cdot \cos\theta$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

Example: Using the Difference Formula for an Exact Value

Find the exact value of $\sin\frac{\pi}{12}$.

$\sin\left(\frac{\pi}{12}\right)$

$\sin\left(\frac{4\pi}{12} - \frac{3\pi}{12}\right)$

$\pi/3$ (α) $\quad$ $\pi/4$ (β)

$= \sin\pi/3 \cdot \cos\pi/4 - \sin\pi/4 \cdot \cos\pi/3$

$=$ ——— ... Do yourself

Find: $\sin\left(\frac{7\pi}{12}\right)$

$\sin\left(\frac{3\pi}{12} + \frac{4\pi}{12}\right)$

$\frac{\pi}{4}$ (α) $\quad$ $\pi/3$ (β)

$= \sin\frac{\pi}{4}\cdot\cos\frac{\pi}{3} + \cos\frac{\pi}{4}\cdot\sin\frac{\pi}{3}$

$= \frac{\sqrt{2}}{2}\cdot\frac{1}{2} + \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}$

$= \frac{\sqrt{2}+\sqrt{6}}{4}$

Example: Recognizing the Sum Formula in Reverse

$\sin(\alpha+\beta) = \sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta$

Find the exact value of

$$\sin 75^\circ \cos 15^\circ + \cos 75^\circ \sin 15^\circ$$

$\alpha = 75^\circ$

$\beta = 15^\circ$

Example: Recognizing the Sum Formula in Reverse

Find the exact value of

$$\sin 75^\circ \cos 15^\circ + \cos 75^\circ \sin 15^\circ$$

Hint: don't evaluate each piece separately — the whole right-hand side of a familiar formula is staring back at you. Which formula, and with $\alpha = 75^\circ$, $\beta = 15^\circ$?

$\sin(\alpha+\beta) = \sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta$.

$\sin 75^\circ \cdot \cos 15^\circ + \cos 75^\circ \cdot \sin 15^\circ$

$= \sin(75^\circ + 15^\circ) = \sin(90^\circ) = 1$

TRY: $\sin 80^\circ \cos 20^\circ - \cos 80^\circ \sin 20^\circ$

$= \ldots\ldots \rightarrow \sin(80^\circ - 20^\circ)$

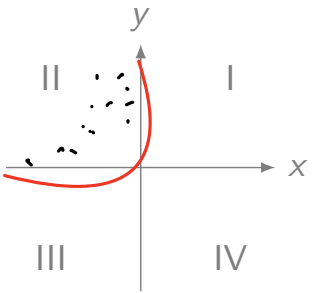
$= \sin 60^\circ = \sqrt{3}/2$.

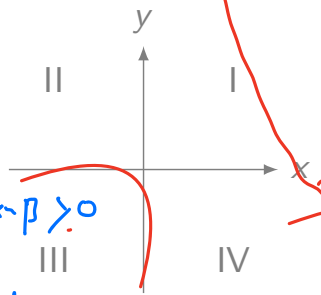
Example: Finding Several Exact Values

Suppose $\sin\alpha = \dfrac{1}{\sqrt{2}}$ with $\dfrac{\pi}{2} < \alpha < \pi$, and $\sin\beta = -\dfrac{1}{3}$ with $\pi < \beta < \dfrac{3\pi}{2}$. Find the exact value of each:

(a) $\cos\alpha$ (b) $\cos\beta$

Special attention: place α and β in their correct quadrants, then use a Pythagorean Identity or a right triangle to recover the missing cosine. Watch the sign!





Example: Finding Several Exact Values

Using the same α and β as before, now find:

(c) $\cos(\alpha+\beta)$ (d) $\sin(\alpha+\beta)$

Special attention: this is simply plugging your four values of $\sin\alpha$, $\cos\alpha$, $\sin\beta$, $\cos\beta$ into the Sum Formulas - no new ideas needed.

$\cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta$

$= \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{2\sqrt{2}}{3}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{3}\right)$

$= \frac{2\sqrt{2}}{\sqrt{2}\cdot 3} + \frac{1}{3\sqrt{2}} = \frac{(2\sqrt{2}+1)}{3\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}(2\sqrt{2}+1)}{6}$

$= \frac{4+\sqrt{2}}{6}$

Theorem: Sum and Difference Formulas for Tangent

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

These follow from dividing the sine sum/difference formula by the cosine sum/difference formula — worth re-deriving once on your own.

$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$

$\tan(\alpha-\beta) = \frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}$

Example: Establishing an Identity

Establish the identity:

$$\frac{\cos(\alpha-\beta)}{\sin\alpha\sin\beta} = \cot\alpha\cot\beta + 1$$

Hint: expand the numerator with the Difference Formula for cosine first, then split the resulting fraction into two pieces.

$$\frac{\cos(\alpha-\beta)}{\sin\alpha\cdot\sin\beta} = \frac{\cos\alpha\cdot\cos\beta + \sin\alpha\cdot\sin\beta}{\sin\alpha\cdot\sin\beta} = \left(\frac{\cos\alpha\cdot\cos\beta}{\sin\alpha\cdot\sin\beta} + \frac{\sin\alpha\cdot\sin\beta}{\sin\alpha\cdot\sin\beta}\right)$$

$$= \cot\alpha\cdot\cot\beta + 1$$

Right side

Example: Establishing an Identity

Establish the identity:

$$\tan(\theta + \pi) = \tan\theta$$

Hint: expand using the Sum Formula for tangent, then evaluate $\tan\pi$ — most of the expression collapses immediately.

Left: $\tan(\theta+\pi) = \dfrac{\tan\theta + \tan\pi}{1 - \tan\theta\cdot\tan\pi}$; $\tan\pi = \dfrac{\sin\pi}{\cos\pi} = 0$

$= \dfrac{\tan\theta + 0}{1 - 0} = \tan\theta$: Right

* This shows $\tan\theta$ is periodic function with period π.

Example: Establishing an Identity

Prove the identity:

$$\tan\left(\theta - \frac{\pi}{2}\right) = -\cot\theta$$

$f(-x) = -f(x)$

Hint: it may be easier to rewrite tan as $\frac{\sin}{\cos}$ and expand numerator and denominator with the Difference Formulas separately.

Left: $\tan(\theta - \pi/2) = \dfrac{\sin(\theta - \pi/2)}{\cos(\theta - \pi/2)} = \dfrac{\sin[-(\pi/2 - \theta)]}{\cos[-(\pi/2 - \theta)]}$ → odd / even

$f(-x) = f(x)$

$= -\dfrac{\sin(\pi/2 - \theta)}{\cos(\pi/2 - \theta)}$

$= -\dfrac{\cos\theta}{\sin\theta} = -\cot\theta$

* $\sin(90° - \theta) = \cos\theta$ ✓

$\cos(90° - \theta) = \sin\theta$ ✓

$\tan(90° - \theta) = \cdots$

Summary: Sum and Difference Formulas

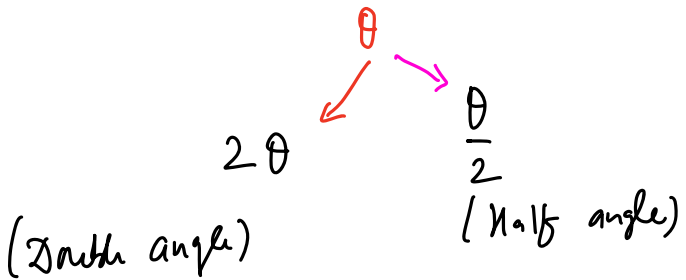
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

Objective 4

Use Double-Angle and Half-Angle Formulas



Theorem: Double-Angle Formulas

$\sin(\alpha+\beta) = \sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta$

If $\alpha = \beta = \theta$

$\sin(2\theta) = \sin\theta\cdot\cos\theta + \cos\theta\cdot\sin\theta$

$= 2\sin\theta\cdot\cos\theta$

$$\sin 2\theta = 2\sin\theta\cdot\cos\theta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

All three forms of $\cos(2\theta)$ are equivalent: pick whichever one matches the information you're given (a known $\sin\theta$ or a known $\cos\theta$).

$\cos(\alpha+\beta) = \cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta$

If $\alpha = \beta = \theta$ $\longrightarrow$ $\cos(2\theta) = \cos\theta\cdot\cos\theta - \sin\theta\cdot\sin\theta$

$= \cos^2\theta - \sin^2\theta \Rightarrow \cos^2\theta - (1-\cos^2\theta) = \cos^2\theta - 1 + \cos^2\theta$

$= 2\cos^2\theta - 1$

Example: Finding Exact Values with the Double-Angle Formulas

If $\sin\theta = \frac{5}{6}$ and $\frac{\pi}{2} < \theta < \pi$, find the exact value of:

(a) $\sin(2\theta)$ (b) $\cos(2\theta)$

$2\sin\theta\cdot\cos\theta$

$\cos^2\theta - \sin^2\theta = (-\frac{1}{2})^2 - (\frac{5}{6})^2 = \cdots$

Special attention: part (a) needs $\cos\theta$ too — find it first (watch the sign in quadrant II) — while part (b) can be done directly from $\sin\theta$ alone using one of the three cosine double-angle forms.

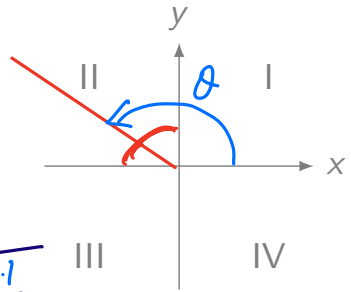
Given: $\sin\theta = \frac{5}{6}$

$\cos\theta = -\sqrt{1-\sin^2\theta}$

$= -\sqrt{1-(\frac{5}{6})^2}$

$= -\sqrt{\frac{36-25}{36}} = -\sqrt{\frac{11}{36}}$

$= -\frac{\sqrt{11}}{6} = -\frac{\sqrt{11}}{6}$



$\sin^2\theta + \cos^2\theta = 1$

$\cos\theta = -\sqrt{1-\sin^2\theta}$

$\sin 2\theta = 2\sin\theta\cdot\cos\theta$

$= 2(\frac{5}{6})\cdot(-\frac{\sqrt{11}}{6})$

$= \frac{-5\sqrt{11}}{18}$.

Example: Deriving New Double-Angle Formulas

(a) Derive a formula for $\tan(2\theta)$ in terms of $\tan\theta$ only.

(b) Derive a formula for $\sin(3\theta)$ in terms of $\sin\theta$ and $\cos\theta$ only.

Special attention: for (a), start from the Sum Formula for tangent with $\alpha = \beta = \theta$. For (b), write $3\theta = 2\theta + \theta$ and apply a Sum Formula together with the Double-Angle Formulas.

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\cdot\tan\beta}$$

$\alpha=\beta=\theta$

$$\tan(2\theta) = \frac{\tan\theta+\tan\theta}{1-\tan\theta\cdot\tan\theta}$$

$$\sin(3\theta) = \sin(2\theta+\theta) = \sin(2\theta)\cos\theta + \cos(2\theta)\cdot\sin\theta$$

$$= 2\sin\theta\cdot\cos\theta\cdot\cos\theta + (\cos^2\theta-\sin^2\theta)\cdot\sin\theta$$

$$\sin(3\theta) = 3\sin\theta\cdot\cos^2\theta - \sin^3\theta$$

$$\sin(\alpha+\beta) = \sin\alpha\cdot\cos\beta + \sin\beta\cdot\cos\alpha$$

Power-Reduction Formulas

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2} \qquad \cos^2\theta = \frac{1+\cos(2\theta)}{2} \qquad \tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$$

These come directly from solving the Double-Angle Formulas for $\cos(2\theta)$ — they trade a squared function for a "flatter" expression, one power lower.

RHS: $\frac{1-\cos(2\theta)}{2} = \frac{1-[1-2\sin^2\theta]}{2}$

$= \frac{1-1+2\sin^2\theta}{2} = \frac{2\sin^2\theta}{2} = \sin^2\theta \rightarrow$ left side.

Example: Eliminating Higher Powers

Rewrite $\cos^4\theta$ as an equivalent expression that involves no power of sine or cosine higher than 1.

Special attention: start by writing $\cos^4\theta = (\cos^2\theta)^2$, then apply a Power-Reduction Formula *twice* — once on the outside, and again on the $\cos^2(2\theta)$ term that appears afterward.

$$\cos^4\theta = (\cos^2\theta)^2 = \left(\frac{1+\cos(2\theta)}{2}\right)^2 = \frac{1}{4}\left[1 + 2\cos(2\theta) + \cos^2(2\theta)\right]$$

$$= \cdots\cdots$$

$$= \cdots\cdots$$

$$= \frac{3}{8} + \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta)$$

Example: Solving a Trigonometric Equation (1 of 2)

Solve: $\sin\theta\cos\theta = -\frac{1}{4}, \quad 0 \le \theta < 2\pi.$

Special attention: the left side is half of a familiar double-angle expression — multiply both sides by 2 to reveal it, then solve for 2θ before dividing back down to θ. Remember: 2θ ranges over $[0, 4\pi)$, so expect more than one family of solutions.

$\rightarrow \quad \sin\theta \cdot \cos\theta = -\frac{1}{4} \qquad (0 \le \theta < 2\pi)$

$\Rightarrow \quad 2 \cdot \sin\theta \cdot \cos\theta = -2 \cdot \frac{1}{4} = -\frac{1}{2}$

$\Rightarrow \quad \sin(2\theta) = -\frac{1}{2}$

$\hookrightarrow \sin\left(\frac{11\pi}{6} + 2\pi k\right)$; k is integer

$\frac{2\theta}{2} = \frac{\frac{11\pi}{6} + 2\pi k}{2} \Rightarrow \theta = \frac{11\pi}{12} + \pi k$

$k = 0, k = 1, k = 2$

$k = -1, k = -2 \ldots$

Soln: $\theta = \frac{11\pi}{12}, \frac{23\pi}{12}$ in interval $[0, 2\pi)$

what about? $\frac{7\pi}{12}, \frac{19\pi}{12}$

Example: Solving a Trigonometric Equation (2 of 2)

Continue solving $\sin(2\theta) = -\frac{1}{2}$ for all $\theta \in [0, 2\pi)$.

Special attention: write the *general* solution for 2θ first (there are two families, k an integer), then substitute $k = 0, 1, 2, \ldots$ into each family and keep only the values of θ that land inside $[0, 2\pi)$.

Try your own!

Theorem: Half-Angle Formulas

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}} \qquad \cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}} \qquad \tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

Special attention: the $\pm$ sign is *not* optional — it must be resolved every time by checking which quadrant $\frac{\alpha}{2}$ actually lies in.

Example: Using a Half-Angle Formula

Use a Half-Angle Formula to find the exact value of:

(a) $\cos 75^\circ$ (b) $\sin(-75^\circ)$ → Do yourself

Ans: $-\frac{\sqrt{2+\sqrt{3}}}{2}$

Special attention: for (a), notice $75^\circ = \frac{150^\circ}{2}$ and decide the sign from the quadrant of 75° itself. For (b), use the odd-function identity $\sin(-75^\circ) = -\sin(75^\circ)$ before reaching for the Half-Angle Formula.

a) $75^\circ = \frac{150^\circ}{2}$ Use half Angle formula:

$$\cos\left(\frac{150^\circ}{2}\right) = \sqrt{\frac{1+\cos 150^\circ}{2}} = \frac{\sqrt{2-\sqrt{3}}}{2}$$

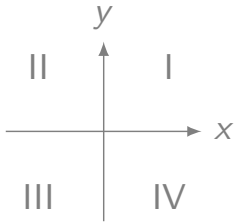
$\cos 150^\circ = -\frac{\sqrt{3}}{2}$

Example: Using a Half-Angle Formula

If $\cos\alpha = -\frac{2}{3}$ with $\pi < \alpha < \frac{3\pi}{2}$, find the exact value of:

(a) $\sin\frac{\alpha}{2}$ (b) $\cos\frac{\alpha}{2}$ (c) $\tan\frac{\alpha}{2}$

Special attention: first pin down which quadrant $\frac{\alpha}{2}$ falls in — that one fact fixes the sign in all three formulas.



$\pi < \alpha < \frac{3\pi}{2}$

$\Rightarrow \frac{\pi}{2} < \frac{\alpha}{2} < \frac{3\pi}{4}$

$\frac{\alpha}{2}$ lies in II quadrant

$\sin\alpha/2 > 0$

$$\sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}} = \sqrt{\frac{1-\left(-\frac{2}{3}\right)}{2}} = \frac{5\sqrt{6}}{6}$$

Try: $\cos\alpha/2$, $\tan\alpha/2$

Reference: An Alternate Tangent Half-Angle Formula

$$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha}$$

Special attention: this version has *no* square root and *no* ambiguous sign — often the fastest route to $\tan\frac{\alpha}{2}$ when both $\sin\alpha$ and $\cos\alpha$ are already known.

Summary

- An *identity* holds for every valid input; always work one side until it matches the other — never cross the equals sign.
- Four moves cover most identities: convert to sine/cosine, multiply by a well-chosen 1, combine fractions, and factor.
- Sum/Difference Formulas turn a non-standard angle into a combination of standard ones — look for $105^\circ = 45^\circ + 60^\circ$ type splits.
- Double-Angle and Half-Angle Formulas trade one angle (θ) for another (2θ or $\frac{\theta}{2}$) — always resolve the $\pm$ sign from the quadrant before finishing a Half-Angle problem.