RECAP:

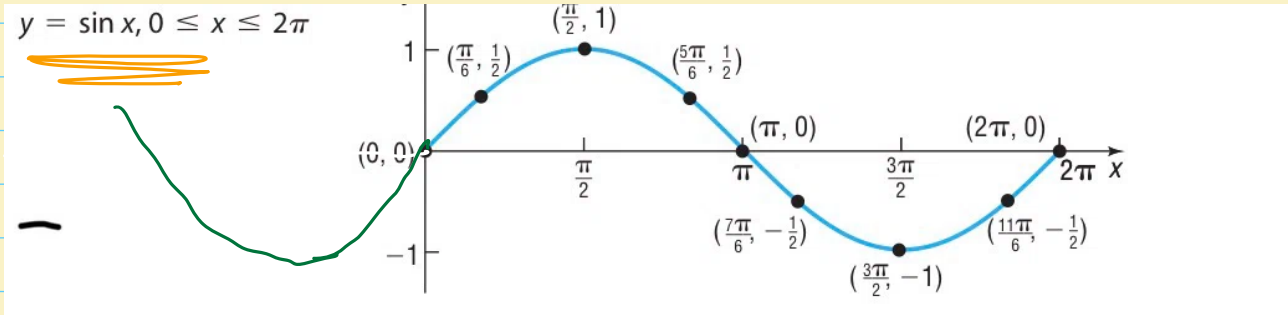
$y = A\sin(\omega x)$, $y = A\cos(\omega x)$

Here: A is the amplitude. The amplitude determines the vertical stretch or compression of the graph of $y = \sin x$ or $y = \cos x$

$|A|$ → the number (amplitude)

* ω → angular frequency

x	$y = \sin x$	(x, y)
0	0	(0, 0)
$\frac{\pi}{6}$	$\frac{1}{2}$	$\left(\frac{\pi}{6}, \frac{1}{2}\right)$
$\frac{\pi}{2}$	1	$\left(\frac{\pi}{2}, 1\right)$
$\frac{5\pi}{6}$	$\frac{1}{2}$	$\left(\frac{5\pi}{6}, \frac{1}{2}\right)$
π	0	$(\pi, 0)$
$\frac{7\pi}{6}$	$-\frac{1}{2}$	$\left(\frac{7\pi}{6}, -\frac{1}{2}\right)$
$\frac{3\pi}{2}$	-1	$\left(\frac{3\pi}{2}, -1\right)$
$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\left(\frac{11\pi}{6}, -\frac{1}{2}\right)$
2π	0	$(2\pi, 0)$



* $f(x)$

$f(x) \longrightarrow f(x) + h$ (vertical shift upward) $h > 0$

$f(x) \longrightarrow f(x) - h$ (vertical shift downward)

$f(x) \rightarrow f(x-h)$ $\quad h>0$

↓ shift horizontally to the right.

$f(x) \rightarrow f(x+h)$ $\quad h>0$

↓ shift horizontally to the left

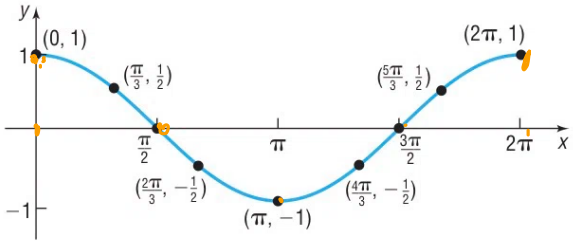
The Graph of the Cosine Function

The cosine function also has period 2π. We proceed as we did with the sine function by constructing Table 7, which lists some points on the graph of $y = \cos x$, $0 \le x \le 2\pi$. As the table shows, the graph of $y = \cos x$, $0 \le x \le 2\pi$, begins at the point $(0, 1)$. As x increases from 0 to $\frac{\pi}{2}$ to π, the value of y decreases from 1 to 0 to -1; as x increases from π to $\frac{3\pi}{2}$ to 2π, the value of y increases from -1 to 0 to 1. As before, plot the points in Table 7 to get one period or cycle of the graph. See Figure 48.

Table 7

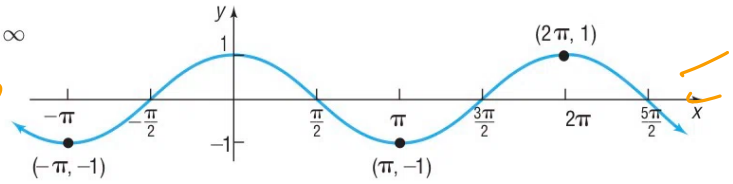
x	$y = \cos x$	(x, y)
0	1	$(0, 1)$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\left(\frac{\pi}{3}, \frac{1}{2}\right)$
$\frac{\pi}{2}$	0	$\left(\frac{\pi}{2}, 0\right)$
$\frac{2\pi}{3}$	$-\frac{1}{2}$	$\left(\frac{2\pi}{3}, -\frac{1}{2}\right)$
π	-1	$(\pi, -1)$
$\frac{4\pi}{3}$	$-\frac{1}{2}$	$\left(\frac{4\pi}{3}, -\frac{1}{2}\right)$
$\frac{3\pi}{2}$	0	$\left(\frac{3\pi}{2}, 0\right)$
$\frac{5\pi}{3}$	$\frac{1}{2}$	$\left(\frac{5\pi}{3}, \frac{1}{2}\right)$
2π	1	$(2\pi, 1)$

Figure 48
$y = \cos x, 0 \le x \le 2\pi$



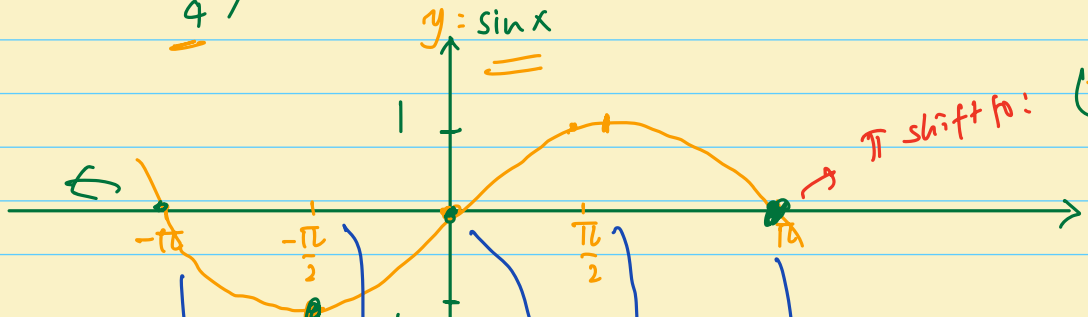
A more complete graph of $y = \cos x$ is obtained by continuing the graph in each direction, as shown in Figure 49.

Figure 49
$y = \cos x, -\infty < x < \infty$



Ex: $y = \sin\left(x - \frac{\pi}{4}\right)$ (Use transformation Rule)

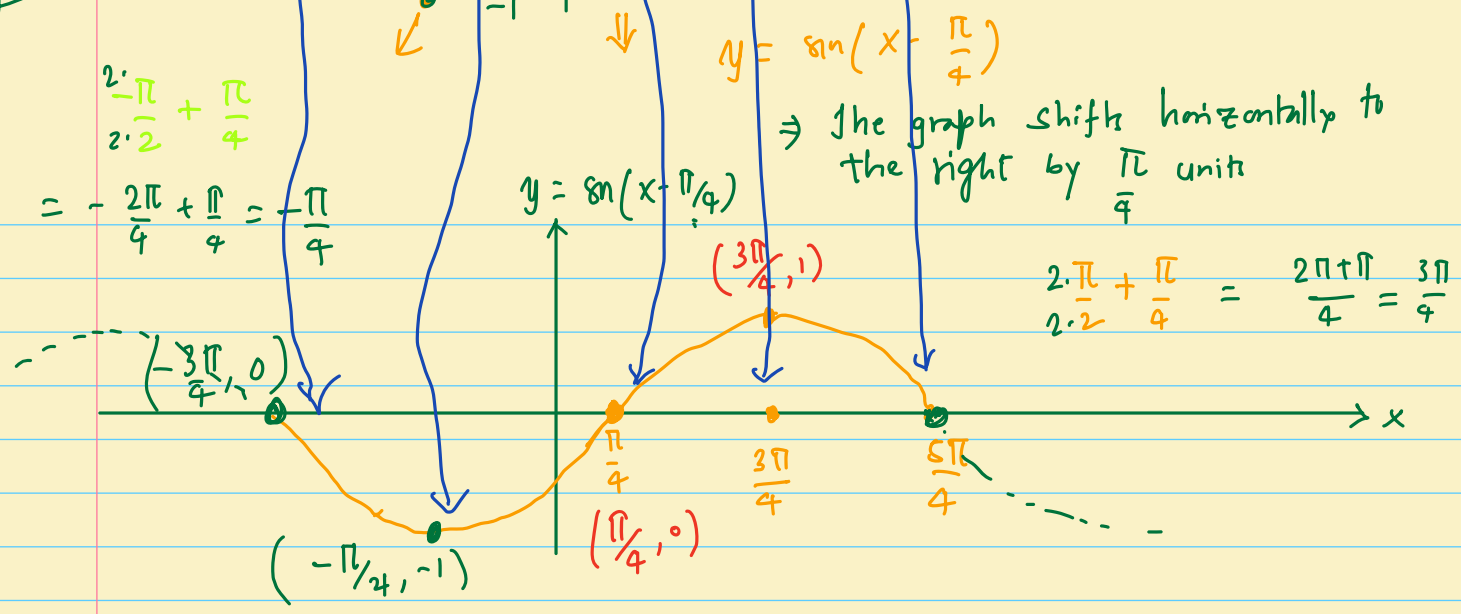
$y = \sin x$



π shift to: $\left(\pi + \frac{\pi}{4}\right) = \left(\frac{5\pi}{4}\right)$

$-\pi + \frac{\pi}{4}$

$= -\frac{3\pi}{4}$



A sinusoidal graph is smooth, wave-like curve that repeats its pattern at regular intervals.

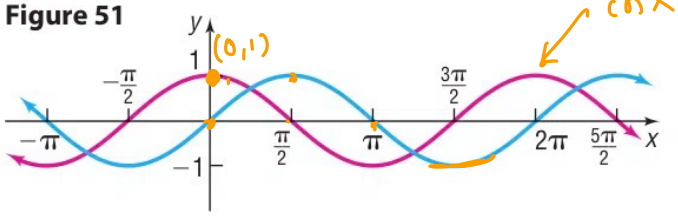
Sinusoidal Graphs

Shift the graph of $y = \cos x$ to the right $\frac{\pi}{2}$ units to obtain the graph of $y = \cos\left(x - \frac{\pi}{2}\right)$. See Figure 51(a). Now look at the graph of $y = \sin x$ in Figure 51(b). We see that the graph of $y = \sin x$ is the same as the graph of $y = \cos\left(x - \frac{\pi}{2}\right)$.

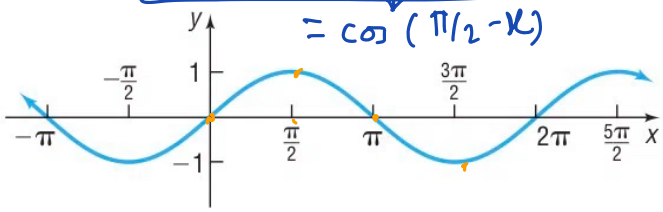
$\sin x = \cos(x - \pi/2)$

$= \cos(\pi/2 - x)$

Figure 51



(a) $y = \cos x$ $y = \cos\left(x - \frac{\pi}{2}\right)$



(b) $y = \sin x$

Let $\omega > 0$ (ω := angular frequency)

The amplitude and period of functions:

$y = A\sin(\omega x)$ & $y = A\cos(\omega x)$

are given by:

Amplitude $= |A|$ and Period $(T) = \frac{2\pi}{\omega}$

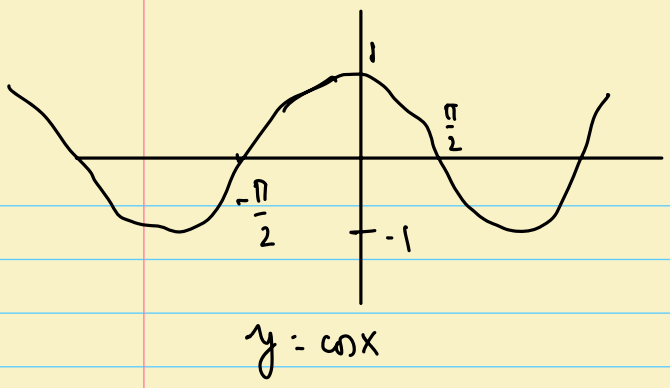
Ex: * If $y = \cos 3x$

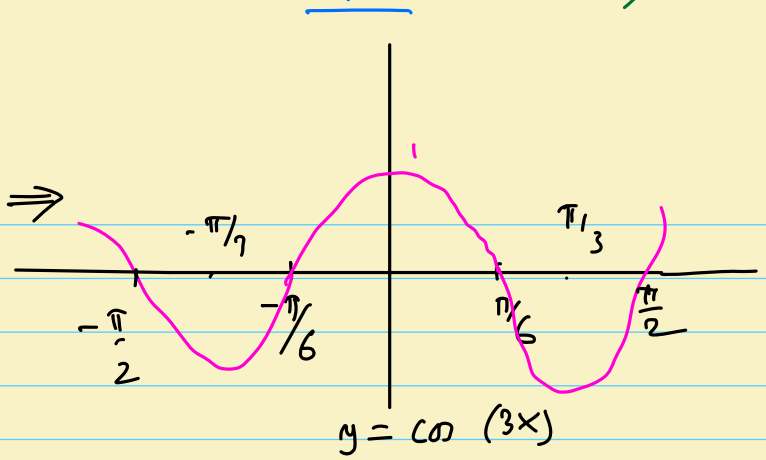
→ Note: $y = f(x)$, $a > 0$

New graph: $y = f(ax)$

Compression (horizontal): $a > 1$

Stretch (horizontal): $a < 1$





THEOREM

If $\omega > 0$, the amplitude and period of $y = A\sin(\omega x)$ and $y = A\cos(\omega x)$ are given by

$$\text{Amplitude} = |A| \qquad \text{Period} = T = \frac{2\pi}{\omega} \qquad (1)$$

Q: If $y = -2\cos(-9x) = -2\cos(9x)$

A

Amplitude $= |A| = |-2| = 2$

$\omega = 9$

Period $= \dfrac{2\pi}{\omega} = \dfrac{2\pi}{9}$

Ex: Graph: $y = 4\sin\left(\frac{1}{4}x\right)$

Hint:

$4 > 0$ → Vertical stretch

$\frac{1}{4} > 0$ → Horizontal stretch by $\left(\frac{1}{4}\right)$

Q: Use graph of $y = \sin x$ to graph

$y = -\sin x + 2$

↙ Reflection about x-axis

↘ shift vertically up by 2 units

Finding the Amplitude and Period of a Sinusoidal Function

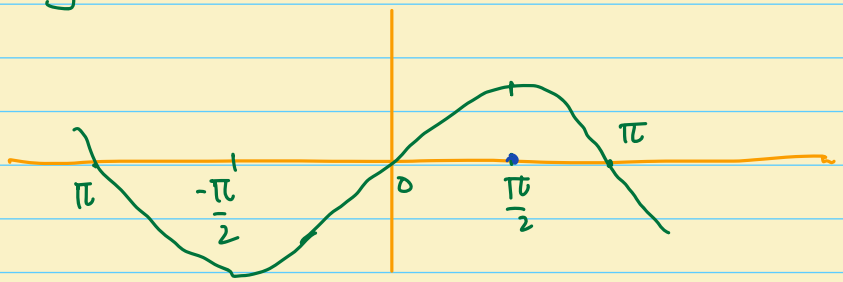
Determine the amplitude and period of $y = 3\sin(4x)$.

Comparing $y = 3\sin(4x)$ to $y = A\sin(\omega x)$, we find that $A = 3$ and $\omega = 4$. From equation (1),

$$\text{Amplitude} = |A| = 3 \qquad \text{Period} = T = \frac{2\pi}{\omega} = \frac{2\pi}{4} = \frac{\pi}{2}$$

* Sketch the graph: $y = -\sin(2x)$

$y = \sin x$:



$y = \sin 2x$

horizontal compression ← $f(ax)$ ($a > 0$)

$f(x) \to a \cdot f(x)$; $a > 1$ I.V. stretch

STEP 1

$y = 3\sin(4x)$

$A = 3$ ↓ $\omega = 4$

$T = \frac{2\pi}{4} = \frac{\pi}{2}$

↓

one cycle will begin at $x = 0$ end at $x = \frac{\pi}{2}$

EXAMPLE 5 How to Graph a Sinusoidal Function Using Key Points

STEP 2 $\left[0, \frac{2\pi}{\omega}\right] = \left[0, \frac{\pi}{2}\right]$ 4 subintervals:

Graph $y = 3\sin(4x)$ using key points.

Step-by-Step Solution

Step 1: Determine the amplitude and period of the sinusoidal function.

Comparing $y = 3\sin(4x)$ to $y = A\sin(\omega x)$, we see that $A = 3$ and $\omega = 4$, so the amplitude is $|A| = 3$ and the period is $\frac{2\pi}{\omega} = \frac{2\pi}{4} = \frac{\pi}{2}$. Because the amplitude is 3, the graph of $y = 3\sin(4x)$ will lie between -3 and 3 on the y-axis. Because the period is $\frac{\pi}{2}$, one cycle will begin at $x = 0$ and end at $x = \frac{\pi}{2}$.

Step 2: Divide the interval $\left[0, \frac{2\pi}{\omega}\right]$ into four subintervals of the same length.

Divide the interval $\left[0, \frac{\pi}{2}\right]$ into four subintervals, each of length $\frac{\pi}{2} \div 4 = \frac{\pi}{8}$ as follows:

$$\left[0, \frac{\pi}{8}\right] \quad \left[\frac{\pi}{8}, \frac{\pi}{8} + \frac{\pi}{8}\right] = \left[\frac{\pi}{8}, \frac{\pi}{4}\right] \quad \left[\frac{\pi}{4}, \frac{\pi}{4} + \frac{\pi}{8}\right] = \left[\frac{\pi}{4}, \frac{3\pi}{8}\right] \quad \left[\frac{3\pi}{8}, \frac{3\pi}{8} + \frac{\pi}{8}\right] = \left[\frac{3\pi}{8}, \frac{\pi}{2}\right]$$

The endpoints of the subintervals are $0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}, \frac{\pi}{2}$. These values represent the x-coordinates of the five key points on the graph.

Step 3: Use the endpoints of these subintervals to obtain five key points on the graph.

COMMENT We could also obtain the five key points by evaluating $y = 3\sin(4x)$ at each value of x. ■

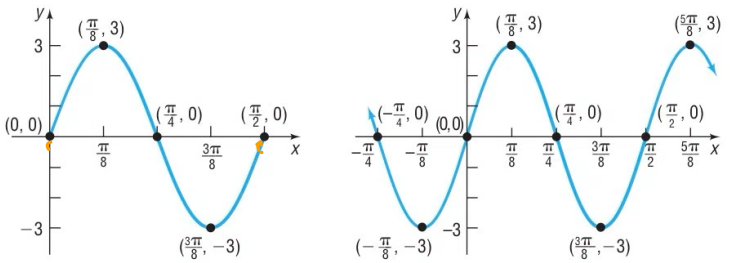
To obtain the y-coordinates of the five key points of $y = 3\sin(4x)$, multiply the y-coordinates of the five key points for $y = \sin x$ in Figure 56(a) by $A = 3$. The five key points are

$$(0, 0) \quad \left(\frac{\pi}{8}, 3\right) \quad \left(\frac{\pi}{4}, 0\right) \quad \left(\frac{3\pi}{8}, -3\right) \quad \left(\frac{\pi}{2}, 0\right)$$

Step 4: Plot the five key points and draw a sinusoidal graph to obtain the graph of one cycle. Extend the graph in each direction to make it complete.

Plot the five key points obtained in Step 3 and fill in the graph of the sine curve as shown in Figure 57(a). Extend the graph in each direction to obtain the complete graph shown in Figure 57(b). Notice that additional key points appear every $\frac{\pi}{8}$ radian.

Figure 57



* $y = A\sin(\omega x - \phi) + B$

$\omega > 0$ $A >$ $\phi > 0$ [shifting towards right]

Amplitude: $= |A|$

Period $= \frac{2\pi}{\omega}$

$A\sin\left[\omega\left(x - \frac{\phi}{\omega}\right)\right] + B$

(phase shift)

Phase shift $= \frac{\phi}{\omega}$

$\phi > 0$ [Right]

$\phi < 0$ [Left]

SUMMARY Steps for Graphing a Sinusoidal Function of the Form $y = A\sin(\omega x)$ or $y = A\cos(\omega x)$ Using Key Points

Step 1: Determine the amplitude and period of the sinusoidal function.

Step 2: Divide the interval $\left[0, \frac{2\pi}{\omega}\right]$ into four subintervals of the same length.

Step 3: Use the endpoints of these subintervals to obtain five key points on the graph.

Step 4: Plot the five key points and draw a sinusoidal graph to obtain the graph of one cycle. Extend the graph in each direction to make it complete.

* $y = 3\sin(2x - \pi)$

$\Rightarrow$ $y = 3\sin\left[2\left(x - \frac{\pi}{2}\right)\right]$

$|A| = 3$ $\quad \omega = 2$ $\quad \frac{\phi}{\omega} = \frac{\pi}{2}$

$T = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi$

EXAMPLE 6 Graphing a Sinusoidal Function Using Key Points

Graph $y = 2\sin\left(-\frac{\pi}{2}x\right)$ using key points.

Solution Since the sine function is odd, we can use the equivalent form:

$$y = -2\sin\left(\frac{\pi}{2}x\right)$$

STEP 1: Comparing $y = -2\sin\left(\frac{\pi}{2}x\right)$ to $y = A\sin(\omega x)$, we find that $A = -2$ and $\omega = \frac{\pi}{2}$. The amplitude is $|A| = |-2| = 2$, and the period is $T = \frac{2\pi}{\omega} = \frac{2\pi}{\frac{\pi}{2}} = 4$.

The graph of $y = -2\sin\left(\frac{\pi}{2}x\right)$ will lie between -2 and 2 on the y-axis. One cycle will begin at $x = 0$ and end at $x = 4$.

STEP 2: Divide the interval $[0, 4]$ into four subintervals, each of length $4 \div 4 = 1$. The x-coordinates of the five key points are

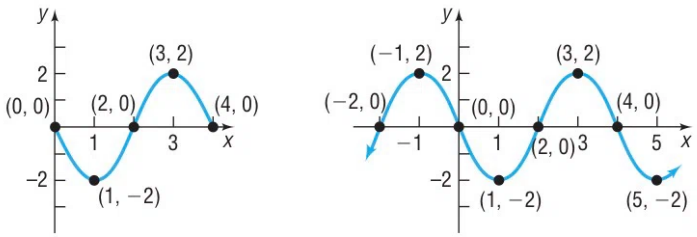
0	$0 + 1 = 1$	$1 + 1 = 2$	$2 + 1 = 3$	$3 + 1 = 4$
1st x-coordinate	2nd x-coordinate	3rd x-coordinate	4th x-coordinate	5th x-coordinate

STEP 3: Since $y = -2\sin\left(\frac{\pi}{2}x\right)$, multiply the y-coordinates of the five key points in Figure 56(a) by $A = -2$. The five key points on the graph are

$$(0, 0) \quad (1, -2) \quad (2, 0) \quad (3, 2) \quad (4, 0)$$

STEP 4: Plot these five points and fill in the graph of the sine function as shown in Figure 58(a). Extend the graph in each direction to obtain Figure 58(b).

Figure 58

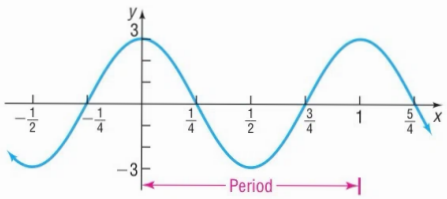


(a) **(b)** $y = 2\sin\left(-\frac{\pi}{2}x\right)$

EXAMPLE 8 Finding an Equation for a Sinusoidal Graph

Find an equation for the graph shown in Figure 60.

Figure 60



Solution The graph has the characteristics of a cosine function. Do you see why? The maximum value, 3, occurs at $x = 0$. So we view the equation as a cosine function $y = A\cos(\omega x)$ with $A = 3$ and period $T = 1$. Then $\frac{2\pi}{\omega} = 1$, so $\omega = 2\pi$. The cosine function whose graph is given in Figure 60 is

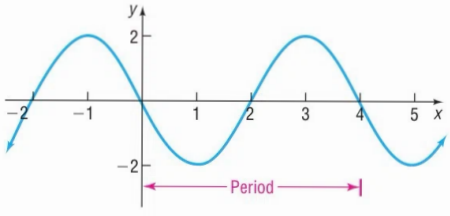
$$y = A\cos(\omega x) = 3\cos(2\pi x)$$

Check: Graph $Y_1 = 3\cos(2\pi x)$ and compare the result with Figure 60.

EXAMPLE 9 Finding an Equation for a Sinusoidal Graph

Find an equation for the graph shown in Figure 61.

Figure 61



Solution The graph is sinusoidal, with amplitude $|A| = 2$. The period is 4, so $\frac{2\pi}{\omega} = 4$ or $\omega = \frac{\pi}{2}$. Since the graph passes through the origin, it is easier to view the equation as a sine function,† but notice that the graph is actually the reflection of a sine function about the x-axis (since the graph is decreasing near the origin). This requires that $A = -2$. The sine function whose graph is given in Figure 61 is

$$y = A\sin(\omega x) = -2\sin\left(\frac{\pi}{2}x\right)$$