# Lecture 4
# Unit Circle Approach; Properties and Graphs of Trigonometric Functions

# Objectives

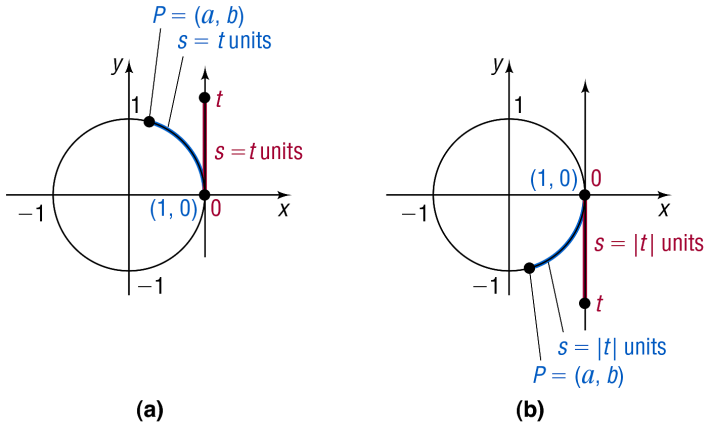
1. Find the Exact Values of the Trigonometric Functions Using the Unit Circle
2. Use Periodic Properties to Find the Exact Values of the Trigonometric Functions
3. Use Even–Odd Properties to Find the Exact Values of the Trigonometric Functions
4. Know the Domain, Range, and Graphs of the Basic Trigonometric Functions

# Objective 1

Find the Exact Values of the Trigonometric Functions Using the Unit Circle

# The Unit Circle Approach

Starting at $(1, 0)$ on the unit circle, travel $s = t$ units counterclockwise (or $s = |t|$ units clockwise if $t < 0$) along the circle to arrive at the point $P = (a, b)$. This is described as wrapping a length of $t$ units around the unit circle.



# Definition: Trigonometric Functions on the Unit Circle

Let $t$ be a real number and let $P = (a, b)$ be the point on the unit circle corresponding to $t$.

$$\sin t = b \qquad \cos t = a$$

$$\tan t = \frac{b}{a} \ (a \neq 0) \qquad \cot t = \frac{a}{b} \ (b \neq 0)$$

$$\csc t = \frac{1}{b} \ (b \neq 0) \qquad \sec t = \frac{1}{a} \ (a \neq 0)$$

# Example 1: Trig Functions from a Point on the Unit Circle

Find $\sin t$, $\cos t$, $\tan t$, $\csc t$, $\sec t$, and $\cot t$ if $P = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ is the point on the unit circle that corresponds to the real number $t$.

$a = -\frac{\sqrt{3}}{2}$, $b = -\frac{1}{2}$

$\sin t = -\frac{1}{2}$

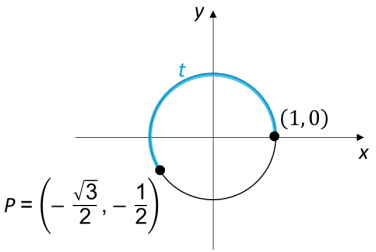
$\cos t = -\frac{\sqrt{3}}{2}$

$\tan t = \frac{-1/2}{-\sqrt{3}/2} = \frac{\sqrt{3}}{3}$

$\csc t = \frac{1}{-1/2} = -2$

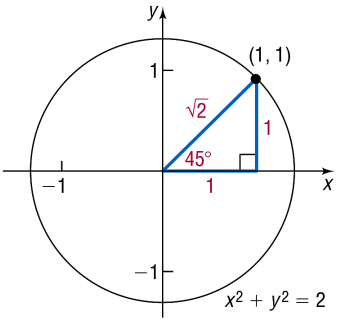
$\sec t = -\frac{2\sqrt{3}}{3}$

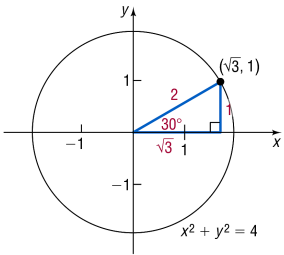
$\cot t = \sqrt{3}$



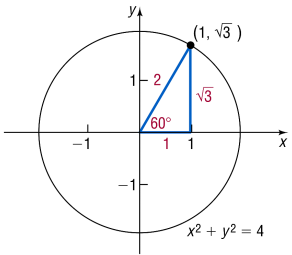
# Key Reference Triangles on the Unit Circle

These special triangles are worth memorizing — they let you read off exact trig values at $30^\circ$, $45^\circ$, and $60^\circ$ directly from the circle.




(a)




(b)

# Trigonometric Functions of a General Angle

If $\theta = t$ radians, the trig functions of $\theta$ equal the trig functions of $t$. More generally, for an angle $\theta$ in standard position, let $P = (a, b)$ be any point — other than the origin — on its terminal side, with $r = \sqrt{a^2 + b^2}$. Then

$$\sin\theta = \frac{b}{r} \qquad \cos\theta = \frac{a}{r} \qquad \tan\theta = \frac{b}{a} \ (a \neq 0)$$

$$\csc\theta = \frac{r}{b} \ (b \neq 0) \qquad \sec\theta = \frac{r}{a} \ (a \neq 0) \qquad \cot\theta = \frac{a}{b} \ (b \neq 0)$$

# Example 2: Trig Functions from a Point on the Terminal Side

The point $P = (-6, 5)$ lies on a circle $x^2 + y^2 = r^2$ that is also on the terminal side of an angle $\theta$ in standard position. Find $\sin\theta$, $\cos\theta$, $\tan\theta$, $\csc\theta$, $\sec\theta$, and $\cot\theta$.

$r = \sqrt{(-6)^2 + 5^2} = \sqrt{36 + 25} = \sqrt{61}, \quad a = -6, \ b = 5$

$$\sin\theta = \frac{5}{\sqrt{61}} = \frac{5\sqrt{61}}{61} \qquad \cos\theta = -\frac{6}{\sqrt{61}} = -\frac{6\sqrt{61}}{61} \qquad \tan\theta = -\frac{5}{6}$$

$$\csc\theta = \frac{\sqrt{61}}{5} \qquad \sec\theta = -\frac{\sqrt{61}}{6} \qquad \cot\theta = -\frac{6}{5}$$

# Objective 2

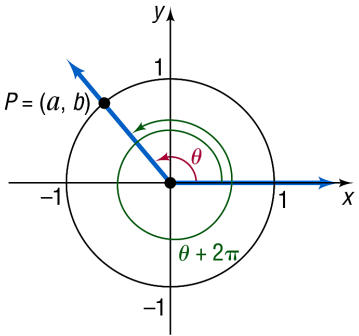
Use Periodic Properties to Find the Exact Values of the Trigonometric Functions

# Periodic Functions

A function $f$ is **periodic** if there is a positive number $p$ such that whenever $\theta$ is in the domain of $f$, so is $\theta + p$, and

$$f(\theta + p) = f(\theta).$$

If there is a smallest such $p$, it is called the **(fundamental) period** of $f$.



# Periodic Properties

$$\sin(\theta + 2\pi k) = \sin\theta \qquad \cos(\theta + 2\pi k) = \cos\theta$$
$$\csc(\theta + 2\pi k) = \csc\theta \qquad \sec(\theta + 2\pi k) = \sec\theta$$
$$\tan(\theta + \pi k) = \tan\theta \qquad \cot(\theta + \pi k) = \cot\theta$$

where $k$ is any integer. (Tangent and cotangent repeat every $\pi$, not just every $2\pi$ — worth pausing on why.)

# Example 3: Using Periodic Properties to Find Exact Values

Find the exact value of each. Do not use a calculator.

$$\text{(a) } \sin(390^\circ) \qquad \text{(b) } \tan(405^\circ) \qquad \text{(c) } \sec\frac{17\pi}{4}$$

(a) $\sin 390^\circ = \sin(30^\circ + 360^\circ) = \sin 30^\circ = \dfrac{1}{2}$

(b) $\tan 405^\circ = \tan(45^\circ + 360^\circ) = \tan 45^\circ = 1$

(c) $\sec\dfrac{17\pi}{4} = \sec\left(\dfrac{\pi}{4} + 4\pi\right) = \sec\dfrac{\pi}{4} = \sqrt{2}$

# Objective 3

Use Even–Odd Properties to Find the Exact Values of the Trigonometric Functions

# Even–Odd Properties

Recall: $f(-x) = f(x)$ means $f$ is **even**; $f(-x) = -f(x)$ means $f$ is **odd**.

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$

$$\csc(-\theta) = -\csc\theta \qquad \sec(-\theta) = \sec\theta \qquad \cot(-\theta) = -\cot\theta$$

# Example 4: Finding Exact Values Using Even–Odd Properties

Find the exact value of each. Do not use a calculator.

$$\text{(a) } \cos(-45^\circ) \qquad \text{(b) } \sec(-45^\circ) \qquad \text{(c) } \csc\left(-\frac{\pi}{6}\right)$$

(a) $\cos(-45^\circ) = \cos 45^\circ = \dfrac{\sqrt{2}}{2}$

(b) $\sec(-45^\circ) = \sec 45^\circ = \sqrt{2}$

(c) $\csc\left(-\dfrac{\pi}{6}\right) = -\csc\dfrac{\pi}{6} = -2$

# Example 5: Combining Periodic and Even–Odd Properties

Find the exact value of each. Do not use a calculator.

$$\text{(a) } \sin(-9\pi) + \cos(6\pi) \qquad \text{(b) } \sec(-7\pi) + \csc\left(-\frac{11\pi}{2}\right)$$

(a) $\sin(-9\pi) + \cos(6\pi) = -\sin(\pi + 8\pi) + \cos(0 + 6\pi) = -\sin\pi + \cos 0 = -0 + 1 = 1$

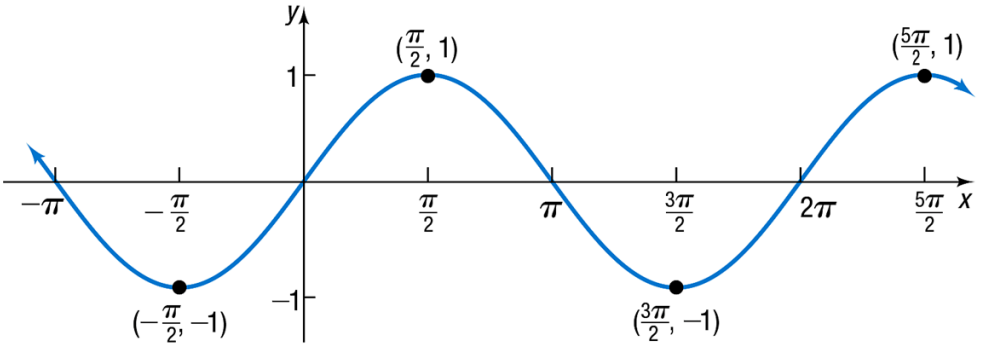
(b) $\sec(-7\pi) + \csc\left(-\frac{11\pi}{2}\right) = \sec(7\pi) - \csc\frac{11\pi}{2} = \sec(\pi + 6\pi) - \csc\left(\frac{3\pi}{2} + 4\pi\right) =$

$\sec\pi - \csc\frac{3\pi}{2} = -1 - (-1) = 0$

## Objective 4

Know the Domain, Range, and Graphs of the Basic Trigonometric Functions

# The Graph of $y = \sin x$

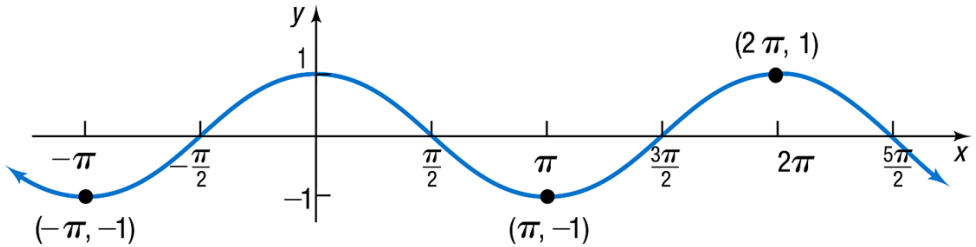



$$y = \sin x, -\infty < x < \infty$$

# Properties of the Sine Function

1. Domain: all real numbers  Range: $[-1, 1]$
2. Sine is an odd function (symmetric about the origin).
3. Period: $2\pi$
4. $x$-intercepts: $\ldots, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$  $y$-intercept: $0$
5. Maximum value $1$ occurs at $x = \ldots, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$
6. Minimum value $-1$ occurs at $x = \ldots, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \ldots$

# The Graph of $y = \cos x$

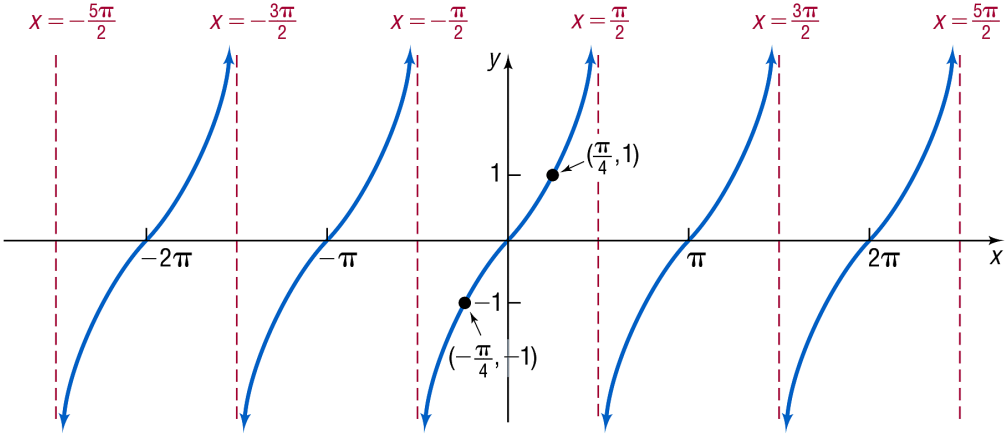



$$y = \cos x, \quad -\infty < x < \infty$$

# Properties of the Cosine Function

1. Domain: all real numbers  Range: $[-1, 1]$
2. Cosine is an even function (symmetric about the $y$-axis).
3. Period: $2\pi$
4. $x$-intercepts: $\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$  $y$-intercept: $1$
5. Maximum value $1$ occurs at $x = \ldots, -2\pi, 0, 2\pi, 4\pi, 6\pi, \ldots$
6. Minimum value $-1$ occurs at $x = \ldots, -\pi, \pi, 3\pi, 5\pi, \ldots$

# The Graph of $y = \tan x$

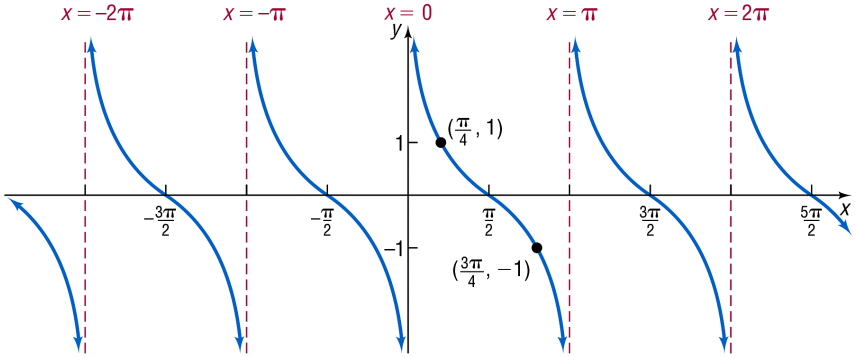



$y = \tan x$, $-\infty < x < \infty$, $x$ not equal to odd multiples of $\frac{\pi}{2}$, $-\infty < y < \infty$

# Properties of the Tangent Function

1. Domain: all reals except odd multiples of $\frac{\pi}{2}$ Range: all real numbers
2. Tangent is an odd function.
3. Period: $\pi$
4. $x$-intercepts: $\ldots, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$ $y$-intercept: $0$
5. Vertical asymptotes occur at $x = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

# The Graph of $y = \cot x$




$y = \cot x$, $-\infty < x < \infty$, $x$ not equal to integer multiples of $\pi$, $-\infty < y < \infty$

# Properties of the Cotangent Function

1. Domain: all reals except integer multiples of $\pi$ Range: all real numbers
2. Cotangent is an odd function.
3. Period: $\pi$
4. Vertical asymptotes occur at $x = \ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$

# Practice: Unit Circle and Terminal-Side Points

**1.** Find $\sin t, \cos t, \tan t, \csc t, \sec t, \cot t$ if $P = \left(\frac{\sqrt{21}}{5}, -\frac{2}{5}\right)$ is on the unit circle at real number $t$.

$\sin t = -\frac{2}{5}$, $\cos t = \frac{\sqrt{21}}{5}$, $\tan t = -\frac{2\sqrt{21}}{21}$, $\csc t = -\frac{5}{2}$, $\sec t = \frac{5\sqrt{21}}{21}$, $\cot t = -\frac{\sqrt{21}}{2}$

**2.** $P = (4, -3)$ is on the circle $x^2 + y^2 = r^2$ and on the terminal side of $\theta$. Find all six trigonometric functions of $\theta$.

$r = \sqrt{16 + 9} = 5$; $\sin\theta = -\frac{3}{5}$, $\cos\theta = \frac{4}{5}$, $\tan\theta = -\frac{3}{4}$, $\csc\theta = -\frac{5}{3}$, $\sec\theta = \frac{5}{4}$, $\cot\theta = -\frac{4}{3}$

# Practice: Periodic and Even–Odd Properties

**3.** Use periodic properties to find the exact value:

$$\text{(a) } \sec\frac{25\pi}{6} \qquad \text{(b) } \csc(450^\circ) \qquad \text{(c) } \sin\frac{9\pi}{4}$$

(a) $\sec\frac{25\pi}{6} = \sec\left(\frac{\pi}{6} + 4\pi\right) = \sec\frac{\pi}{6} = \frac{2\sqrt{3}}{3}$ (b) $\csc(450^\circ) = \csc(90^\circ + 360^\circ) = \csc 90^\circ = 1$ (c) $\sin\frac{9\pi}{4} = \sin\left(\frac{\pi}{4} + 2\pi\right) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

**4.** Use even–odd properties to find the exact value:

$$\text{(a) } \cos(-30^\circ) \qquad \text{(b) } \tan(-30^\circ) \qquad \text{(c) } \sec\left(-\frac{\pi}{6}\right)$$

(a) $\cos(-30^\circ) = \cos 30^\circ = \frac{\sqrt{3}}{2}$ (b) $\tan(-30^\circ) = -\tan 30^\circ = -\frac{\sqrt{3}}{3}$ (c) $\sec\left(-\frac{\pi}{6}\right) = \sec\frac{\pi}{6} = \frac{2\sqrt{3}}{3}$